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ON THE PRODUCT $(ax+by+c)(dx+ey)$

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By

P. Kanagasabapathy, *Ceylon*

(Communicated by the Secretary—Received December, 28, 1957)

1. Let $ax+by$ and $dx+ey$ be two homogeneous linear forms in x, y of determinant 1, with real coefficients, and suppose that neither form represents zero for integers x, y not both zero. Let c be any real number, not zero. In a paper published sometime ago (Kanagasabapathy, 1952), I proved that there exist integers x, y not both zero, such that

$$|(ax+by+c)(dx+ey)|<\frac{1}{4{\cdot}25}.$$

In this paper, I prove that the result is still true if the number 4.25 is replaced by 4.25777.

2 To prove this result, we need the following lemma.

Lemma (Davenport, 1952). *Let $ax+by$ and $dx+ey$ be any two linear forms in x, y of determinant 1, neither of which represents zero for integers x, y not both zero; and let c be any real number other then zero. Then there exists an integral unimodular transformation from x, y to new variables u, v which transforms $(ax+by+c)(dx+ey)$ into*

$$\pm\frac{1}{\theta+\varphi}(u+\theta v-\alpha)(u-\varphi v)$$

where $0<\varphi<1,\quad 1\leq\alpha<\theta.$ (1)

I now establish the following theorem.

Theorem. *Let $ax+by+c$ and $dx+ey$ be two forms as already defined. Then there exist integers x, y not both zero, such that*

$$|(ax+by+c)(dx+ey)|<\frac{1}{4{\cdot}25777}.$$

Proof. By the Lemma, it is sufficient to show that, if θ, φ, α satisfy (1), then there exists integers u, v not both zero such that

$$|(u+\theta v-\alpha)(u-\varphi v)|<\frac{\theta+\varphi}{4{\cdot}25777}.$$

We assume the contrary, namely that

$$|u+\theta v-\alpha)(u-\varphi v)| \geq \frac{\theta+\varphi}{4{\cdot}25777} \tag{2}$$

for all integers u, v not both zero, and deduce a contradiction.

We shall make frequent use of three cases of (2), namely

$$(1+\theta-\alpha)(1-\varphi) \geq \frac{\theta+\varphi}{4{\cdot}25777}, \tag{3}$$

$$(\theta-\alpha)\varphi \geq \frac{\theta+\varphi}{4{\cdot}25777}, \tag{4}$$

$$(1+2\theta-\alpha)\,|1-2\varphi| \geq \frac{\theta+\varphi}{4{\cdot}25777}. \tag{5}$$

Let $n < \alpha < n+1$, where n is a positive integer. Then from (2) we obtain

$$\alpha-n \geq \frac{\theta+\varphi}{4{\cdot}25777n} \tag{6}$$

and

$$n+1-\alpha \geq \frac{\theta+\varphi}{4{\cdot}25777(n+1)} \tag{7}$$

Hence, by addition,

$$\theta+\varphi \leq \frac{n(n+1)}{2n+1}\, 4{\cdot}25777 \tag{8}$$

From (3) and (6) we obtain

$$1-\varphi \geq \frac{\theta+\varphi}{4{\cdot}25777(\theta+\varphi-\varphi-n+1)-(\theta+\varphi)/n} \tag{9}$$

Since the right-hand side decreases as $\theta+\varphi$ increases, we can replace $\theta+\varphi$ by its upper bound from (8), obtaining

$$1-\varphi \geq \frac{n(n+1)}{4{\cdot}25777n(n+1)-(2n+1)\varphi-2n^2}. \tag{10}$$

Similarly, from (5), (6), (8) we obtain

$$|2\varphi-1| \geq \frac{n(n+1)}{8{\cdot}51554n(n+1)-2(2n+1)\varphi-2n^2}; \tag{11}$$

and from (4), (6), (8) we obtain

$$\varphi \geq \frac{n(n+1)}{4.25777n(n+1)-(2n+1)\varphi-2n^2-2n-1}. \tag{12}$$

Case 1. Suppose first that $\varphi < \frac{1}{2}$. Combining (11) and (12), we get

$$1 \geq \frac{n(n+1)}{8{\cdot}51554n(n+1)-2(2n+1)\varphi-2n^2}+\frac{2n(n+1)}{4{\cdot}25777n(n+1)-(2n+1)\varphi-2n^2-n-1}\,.$$

Since the right-hand side increases with φ, and $\varphi > 0$, this implies

$$1 > \frac{1}{8{\cdot}51554-2/n(n+1)}+\frac{2}{2{\cdot}25777-1/n(n+1)}$$

$$> \frac{1}{7{\cdot}51554}+\frac{2}{2{\cdot}25777}\text{, for all } n,$$

$$> 0{\cdot}133+0{\cdot}885$$

and we have a contradiction.

Case 2. Suppose next that $\varphi \geq \frac{1}{2}$. Then, by (11),

$$2\varphi-1 \geq \frac{n(n+1)}{8.51554n(n+1)-2(2n+1)\varphi-2n^2}.$$

Since the right-hand side increases with φ, and $\varphi \geq \frac{1}{2}$, we have

$$2\varphi-1 \geq \frac{1}{65{\cdot}1554-1/n(n+1)}$$

$$> \frac{1}{6{\cdot}51554}\text{, for all } n.$$

Hence $$\varphi > 0{\cdot}5767 \tag{13}$$

Similarly, from (10) we obtain

$$1-\varphi > \frac{1}{2{\cdot}25777+(2n-1)/2n(n+1)} \tag{14}$$

$$\geq \frac{1}{2{\cdot}50777}\text{, for all } n,$$

$$> 0{\cdot}398.$$

Hence $$\varphi < 0{\cdot}602. \tag{15}$$

Further, by combining (4) and (6), we get

$$\theta-n \geq \frac{(n+\varphi)(\theta+\varphi)}{4{\cdot}25777n\varphi}.$$

Hence, writing $\theta-n$ as $(\theta+\varphi)-(n+\varphi)$, we obtain

$$\theta+\varphi \geq \left[\frac{1}{n+\varphi}-\frac{1}{4{\cdot}25777n\varphi}\right]^{-1}. \tag{16}$$

Again, if $\theta-\alpha > m$, then, by (2),

$$\theta-\alpha-m \geq \frac{\theta+\varphi}{4{\cdot}25777(m+\varphi)},$$

Combining this inequality with (6), we obtain

$$\theta-(m+n) \geq \frac{(\theta+\varphi)(m+n+\varphi)}{4{\cdot}25777n(m+\varphi)}.$$

Hence
$$\theta+\varphi \geq \left[\frac{1}{m+n+\varphi} - \frac{1}{4{\cdot}25777n(m+\varphi)}\right]^{-1}. \tag{17}$$

Similarly, if $\theta-\alpha < l$, using (2) and (7), we obtain

$$\theta+\varphi \leq \left[\frac{1}{l+n+1+\varphi} + \frac{1}{4{\cdot}25777(n+1)(l+\varphi)}\right]^{-1}. \tag{18}$$

We now consider the different values of n.

(*i*) *Let* $n=1$. By (12) we have

$$\begin{aligned}\varphi &\geq \frac{2}{8{\cdot}51554-3\varphi-5}\\ &> \frac{2}{8{\cdot}51554-\frac{3}{2}-5}, \text{ since } \varphi > \tfrac{1}{2},\\ &> 0{\cdot}9\end{aligned}$$

which contradicts (15).

(*ii*) *Let* $n=2$. By (12) we have

$$\begin{aligned}\varphi &\geq \frac{6}{12{\cdot}54662-5\varphi}\\ &> \frac{6}{12{\cdot}54662-5(0{\cdot}5767)}, \text{ since } \varphi > 0.5767\\ &> 0{\cdot}62\end{aligned}$$

which contradicts (15).

(*iii*) *Let* $n=3$. By (8) we have

$$\theta+\varphi \leq \tfrac{12}{7}(4{\cdot}25777)$$

Hence
$$\theta+\varphi < 7{\cdot}3 \tag{19}$$

By (10), we have
$$\begin{aligned}1-\varphi &\geq \frac{12}{33{\cdot}09324-7\varphi}\\ &> \frac{12}{33{\cdot}09324-7(0{\cdot}5767)}, \text{ since } \varphi > 0.5767,\\ &> 0{\cdot}4128\end{aligned}$$

Hence $\varphi < 0\cdot5872$.

By (16) with $n = 3$,

$$\theta+\varphi \geq \left[\frac{1}{3+\varphi} - \frac{1}{12\cdot77331\varphi}\right]^{-1}.$$

Since the right-hand side decreases as φ increases and since $\varphi < 0.588$, we have

$$\theta+\varphi > \left[\frac{1}{3\cdot588} - \frac{1}{(12\cdot77331)(0\cdot588)}\right]^{-1}.$$

Hence $$\theta+\varphi > 6\cdot8728. \tag{20}$$

Suppose that $\theta-\alpha > 3$. By (17), with $n = 3$ and $m = 3$, we have

$$\theta+\varphi \geq \left[\frac{1}{6+\varphi} - \frac{1}{12\cdot77331(3+\varphi)}\right]^{-1}.$$

Since the right-hand side increases with φ and $\varphi > 0\cdot576$, we have

$$\theta+\varphi > \left[\frac{1}{6\cdot576} - \frac{1}{(12\cdot77331)(3\cdot576)}\right]^{-1}.$$

$$> 7.68$$

which contradicts (19). Hence $\theta-\alpha < 3$.

By (18), with $n = 3$ and $l = 3$, we have

$$\theta+\varphi \leq \left[\frac{1}{7+\varphi} + \frac{1}{17\cdot03108(3+\varphi)}\right]^{-1}.$$

Since the right-hand side increases with φ and $\varphi < 0\cdot588$, we get

$$\theta+\varphi < \left[\frac{1}{7\cdot588} + \frac{1}{(17\cdot03108)(3\cdot588)}\right]^{-1}.$$

$$< 6\cdot753,$$

which contradicts (20).

(*iv*) *Let* $n = 4$. By (8) we have

$$\theta+\varphi \leq \tfrac{20}{9}(4\cdot25777) < 9\cdot462$$

Suppose that $\theta-\alpha \geq 4$. Then, since $\varphi > 0\cdot57$, we have, by (17)

$$\theta+\varphi > \left[\frac{1}{8\cdot57} - \frac{1}{(17\cdot03108)(4\cdot57)}\right]^{-1}$$

$$> 9\cdot63,$$

which contradicts (21). Hence $\theta-\alpha < 4$.

By (18) with $n = 4$, $l = 4$, and from the fact that $\varphi < 0\cdot61$, we obtain

$$\theta+\varphi < \left[\frac{1}{9\cdot61} + \frac{1}{(21\cdot28885)(4\cdot61)}\right]^{-1}$$

$$< 8\cdot757.$$

Now, by (9),
$$1-\varphi \geq \frac{\theta+\varphi}{4{\cdot}25777(\theta+\varphi-\varphi-3)-\frac{1}{4}(\theta+\varphi)},$$

Since the right-hand side decreases as $\theta+\varphi$ increases, and $\theta+\varphi < 8{\cdot}757$, we have

$$1-\varphi > \frac{8.757}{4{\cdot}25777(5{\cdot}757-\varphi)-2{\cdot}189}$$

Again, since the right-hand side decreases as φ decreases, and $\varphi > 0{\cdot}576$, we have

$$1-\varphi > \frac{8{\cdot}757}{(4{\cdot}25777)(5{\cdot}181)-2{\cdot}189}$$
$$> 0{\cdot}44.$$

Hence $\varphi < 0.56$, which contradicts (13).

(*v*) *Let* $n = 5$. By (8) we have

$$\theta+\varphi \leq \tfrac{30}{11}(4{\cdot}25777) < 11{\cdot}6121 \qquad (22$$

By (11), since $\varphi > 0{\cdot}5767$, we have

$$2\varphi-1 > \frac{30}{205{\cdot}4662-22(0{\cdot}5767)}$$
$$> 0{\cdot}15559.$$

Hence
$$\varphi > 0{\cdot}57779.$$

Once again, by (11), since $\varphi \geq 0{\cdot}57779$, we have

$$2\varphi-1 \geq \frac{30}{205{\cdot}4662-22(0{\cdot}57779)}$$
$$> 0{\cdot}1556.$$

Now, suppose that $\theta-\alpha \geq 5$. By (17), since $\varphi > 0{\cdot}5778$, we have

$$\theta+\varphi > \left[\frac{1}{10{\cdot}5778}-\frac{1}{(21{\cdot}28885)(5{\cdot}5778)}\right]^{-1}$$
$$\geq 11{\cdot}6122$$

which contradicts (22). Hence $\theta-\alpha < 5$.

By (18) with $l = n = 5$ and from the fact that $\theta < 0{\cdot}61$, we have

$$\theta+\varphi < \left[\frac{1}{11{\cdot}61}+\frac{1}{(25{\cdot}54662)(5{\cdot}61)}\right]^{-1}$$
$$< 10.75.$$

Considering (9), and using the fact that $\theta+\varphi < 10{\cdot}75$ and $\varphi > 0{\cdot}577$, we obtain

$$1-\varphi \geq \frac{10{\cdot}75}{(4{\cdot}25777)(6{\cdot}173)-2{\cdot}15}$$
$$\geq 0{\cdot}44.$$

Hence $\varphi < 0\cdot 56$ which contradicts (13).

(*vi*) *Let* $n = 6$. By (11), since $\varphi > 0\cdot 5767$, we have

$$2\varphi-1>\frac{42}{285\cdot 65268-26(0\cdot 5767)}$$

$$>0\ 1551.$$

Hence $$\varphi > 0.5775. \tag{23}$$

By (10), since $\varphi > 0.5775$, we have

$$1-\varphi>\frac{42}{106\cdot 82634-18(0\cdot 5775)}$$

$$>0\cdot 4228.$$

Hence $\varphi < 0\cdot 5772$, which contradicts (23).

(*vii*) *Let* $n = 7$. By (10), since $\varphi > 0\cdot 5767$, we have

$$1-\varphi>\frac{56}{140\cdot 48512-15(0\cdot 5767)}$$

$$>0\cdot 4248.$$

Hence $\varphi < 0\cdot 5752$, which contradicts (13).

(*viii*) *Let* $n \geq 8$. By (14), we have

$$1-\varphi>\frac{1}{2\cdot 25777+(2n-1)/2n(n+1)}\,.$$

Since the right-hand side increases with n, and $n \geq 8$, we have

$$1-\varphi>\frac{1}{2\cdot 25777+15/144}>0\cdot 42337.$$

Hence $\varphi < 0\cdot 57663$, which contradicts (13).

The proof of the Theorem is now complete.

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ON LAPLACE TRANSFORM

By

A. N. Mehra, *Lucknow, U. P.*

(*Communicated by Dr. S. K Bose—Received January 7, 1958.*)

1. The infinite integral

$$p\int_0^\infty e^{-pt}f(t)dt = \varphi(p) \tag{1.1}$$

where p is a suitable parametar, plays an important part in the study of the properties of Functions of Analysis. The integral relation (1.1) between $\varphi(p)$, and $f(t)$, following Van der Pol, is denoted symbolically as

$$\varphi(p) \doteqdot f(t) \tag{1.2}$$

$\varphi(p)$ is called the *image* of $f(t)$ and $f(t)$ the *original* of $\varphi(p)$ and $\varphi(p)$ is known as the *Laplace transform* of $f(t)$.

When $\varphi(p)$ is known and $\varphi(p)/p$ is an analytic function of the complex variable p, then, by means of Bromwich—Wagner theorem, we get back $f(t)$ as

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}\varphi(p).\frac{dp}{p} \tag{1.3}$$

which follows from the Mellin-Fourier theorem.

The object of this paper is to study some of the properties of a *chain* of Laplace transforms. The main results are given in the form of theorems. A few particular cases have been given and they have been illustrated by suitable examples.

2. **Theorem I.** *If* $\varphi(p) \doteqdot f(t), \qquad (p^{\frac{1}{2}})f(1/p) \doteqdot g(t),$

$$\psi(p) \doteqdot \chi(t) \quad \textit{and} \quad (p^{\frac{1}{2}})\chi(1/p) \doteqdot F(t),$$

then

$$\int_0^\infty \varphi(U^2)F\left(\frac{U^2}{4}\right)dU = \int_0^\infty \psi(V^2)g\left(\frac{V^2}{4}\right)dV \tag{2.1}$$

provided the integrals converge absolutely, the Laplace transforms of $|tg(t^2/4)|$ *and* $|tF(t^2/4)|$ *exist, and also the integrals* $\int_0^\infty e^{-t/x}|g(t)|\,dt$ *and* $\int_0^\infty e^{-t/x}|F(t)|\,dt$ *converge.*

Proof. If $\qquad \varphi(p) \doteqdot f(t), \quad (p^{\frac{1}{2}})f(1/p) \doteqdot g(t),$

then (McLachlan and Humbert, 1941, p. 13)*, we have

$$\varphi(p^2) \doteqdot \tfrac{1}{2}\pi^{\frac{1}{2}}tg(\tfrac{1}{4}t^2)$$

---

* We shall denote this book as $M(1)$ in this paper.

Similarly, $\psi(p^2) \doteqdot \frac{1}{2}\pi^{\frac{1}{2}} t F(\frac{1}{2}t^2)$

Now, the result follows on using the following result (Goldstein, 1932, p. 105):

If $\varphi_1(p) \doteqdot f_1(t), \quad \varphi_2(p) \doteqdot f_2(t)$, then

$$\int_0^\infty \varphi_1(U) f_2(U)\, \frac{dU}{U} = \int_0^\infty \varphi_2(V) f_1(V)\, \frac{dV}{V}\,. \tag{A}$$

Particular Cases. (1) Let ($M_{(1)}$, p. 39)

$$\begin{aligned} g(t) &= K_{\pm\frac{1}{4}}(t) \\ &\doteqdot \frac{\pi p}{2^{\frac{1}{2}}(p+1)^{\frac{1}{2}}}, \end{aligned}$$

therefore ($M_{(1)}$, p. 14)

$$f(t) = \frac{\pi}{2^{\frac{1}{2}}(t+1)^{\frac{1}{2}}} \doteqdot \frac{\pi^{3/2}}{2^{\frac{1}{2}}}\, p^{\frac{1}{2}} \operatorname{erfc} p^{\frac{1}{2}} = \varphi(p).$$

Hence from Theorem I, we obtain

$$\frac{\pi^{3/2}}{2^{\frac{1}{2}}} \int_0^\infty t\, e^{t^2} \operatorname{erfc}(t)\, F(\tfrac{1}{2}t^2)\,dt = \int_0^\infty \psi(t^2)\, K_{\pm\frac{1}{4}}(\tfrac{1}{4}t^2)\, dt \tag{2.2}$$

provided the integrals converge absolutely, the Laplace transform of $|tF(\frac{1}{2}t^2)|$ exists and the integral $\int_0^\infty e^{-t/x}|F(t)|\,dt$ is absolutely convergent.

(2) Let ($M_{(1)}$, p. 28)

$$\begin{aligned} g(t) &= t^{\frac{1}{2}\nu} J_\nu(2a(t^{\frac{1}{2}})) \\ &\doteqdot a^\nu p^{-\nu} e^{-a^2/p}, \quad R(\nu+\tfrac{1}{2}) > 0 \text{ and } R(p) > 0. \end{aligned}$$

Therefore ($M_{(1)}$, p. 16)

$$f(t) = a^\nu t^{\nu+\frac{1}{2}} e^{-a^2 t} \doteqdot \frac{a^\nu \Gamma(\nu+\frac{3}{2})p}{(p+a^2)^{\nu+3/2}} = \varphi(p),$$

provided $\mathbf{R}(\nu+\frac{3}{2}) > 0$ and $\mathbf{R}(p+a^2) > 0$.

Hence from Theorem I, we get

$$\int_0^\infty \frac{U^3}{(U^2+a^2)^{\nu+3/2}} F(\tfrac{1}{4}U^2)\,dU = \frac{1}{(2a)^\nu \Gamma(\nu+\frac{3}{2})} \int_0^\infty t^\nu J_\nu(at)\psi(t^2)\,dt \tag{2.3}$$

provided $\mathbf{R}(\nu+\frac{3}{2}) > 0$ and under the conditions of case (1).

Further, if we take $t^{\nu-\frac{1}{2}}\psi(t^2)$ to be *self reciprocal* in Hankel Transform of order ν, then

$$\int_0^\infty \frac{U^2}{(U^4+a^2)^{\nu+3/2}} F(\tfrac{1}{4}U^2)\,dU = \frac{1}{2^\nu a \Gamma(\nu+\frac{3}{2})}\, \psi(a^2) \tag{2.4}$$

under the conditions of (2.3).

Examples. (1) Let ($M_{(1)}$, p. 16)

$$F(t) = e^{-2t} \doteqdot p/p+2,$$

therefore (McLachlan, Humbert and Poli, 1950, p. 23)

$$\chi(t) = t^{\frac{1}{2}}/1+2t \doteqdot \tfrac{1}{2}((\pi p)^{\frac{1}{2}})e^{\frac{1}{4}p} D_{-2}(p^{\frac{1}{2}}) = \psi(p).$$

Further, using the known integral (Goldstein, *loc. cit.*, p. 117), we have

$$\frac{\pi^{3/2}}{2^{\frac{1}{2}}} \int_0^\infty t\, e^{\frac{1}{2}t^2} \operatorname{erfc}(t)\, dt = \frac{\pi^{3/2}}{4(2^{\frac{1}{2}})^2}\, {}_2F_1\left[\begin{matrix} \frac{3}{2}, 1 \\ 2 \end{matrix} ; \tfrac{1}{2}\right]$$

Hence ‡, from (2 2), we have

$$\int_0^\infty t\, e^{\frac{1}{4}t^2} D_{-2}(t) K_{\pm\frac{1}{4}}(\tfrac{1}{4}t^2)\, dt = \frac{\pi}{2(2^{\frac{1}{2}})^2}\, {}_2F_1\left[\begin{matrix} \frac{3}{2}, 1 \\ 2 \end{matrix} ; \tfrac{1}{2}\right]$$

$$= 2^{\frac{1}{2}}(2^{\frac{1}{2}} - 1)\pi \tag{2.5}$$

(2) Let ($M_{(1)}$, p. 20)

$$F(t) = \sin 2(t^{\frac{1}{2}}) \doteqdot (\pi/p)^{\frac{1}{2}} e^{-1/p},$$

therefore ($M_{(1)}$, p. 16)

$$\chi(t) = \pi^{\frac{1}{2}}\, t\, e^{-t} \doteqdot \frac{(\pi^{\frac{1}{2}})p}{(p+1)^2} = \psi(p).$$

Substituting in (2.3) we get

$$\int_0^\infty \frac{U^2 \sin U}{(U^2+a^2)^{\nu+3/2}}\, dU = \frac{\pi^{\frac{1}{2}}}{(2a)^\nu \Gamma(\nu+\frac{3}{2})} \int_0^\infty \frac{t^{\nu+2} J_\nu(at)}{(t^2+1)^2}\, dt.$$

Now, evaluating the integral on the R.H.S by a known integral (Bateman Project, vol 1, 1954, 327)**

$$\int_0^\infty \frac{U^2 \sin U\, dU}{(U^2+a^2)^{\nu+3/2}} = C\left[\frac{a^\nu B(\nu+\frac{3}{2}, \frac{1}{2}-\nu)}{2^{\nu+1}\Gamma(\nu+1)}\, {}_1F_2\left(\begin{matrix} \nu+\frac{3}{2} ; \\ \nu+\frac{1}{2}, \nu+1 ; \end{matrix} \frac{a^2}{4}\right)\right.$$

$$\left. + \frac{a^{1-\nu}\Gamma(\nu-\frac{1}{2})}{2^{1-\nu}\pi^{\frac{1}{2}}}\, {}_1F_2\left(\begin{matrix} 2 \\ \frac{3}{2}, \frac{5}{2}, \end{matrix} ; \frac{a^2}{4}\right)\right] \tag{2.6}$$

where $C = \pi^{\frac{1}{2}}/(2a)^\nu \Gamma(\nu+\frac{3}{2})$, $a > 0$ and $-\frac{1}{2} < R(\nu) < \frac{5}{2}$

3 **Theorem II.** *If* $\varphi(p) \doteqdot f(t)$,

and $\psi(p) \doteqdot 1/t^{\frac{1}{2}} f\, 2(t^{\frac{1}{2}}))$,

[illegible] to ma[illegible]

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* We shall denote this book as $M_{(2)}$ in this paper

‡ The last step follows from

[illegible]

$${}_2F_1\left[\begin{matrix} \frac{1}{2}+a, 1+a \\ 1+2a \end{matrix} ; -x\right] = (1+x)^{-\frac{1}{2}}[2x^{-1}\{((1+x)^{\frac{1}{2}}) - 1\}]^{2a} \text{ with } a = \tfrac{1}{2} \text{ and } x = -\tfrac{1}{2}.$$

** We shall denote this book by $B_{(1)}$.

then
$$\frac{1}{\pi^{\frac12}}\int_0^\infty V^{-3/2}\varphi(1/V)\psi(V)\,dV=\int_0^\infty U^{-1}\varphi(U^{\frac12})f(U)\,dU, \tag{3.1}$$

provided the integrals are absolutely convergent, and the Laplace transforms of $|1/t^{\frac12}f(2(t^{\frac12}))|$ *and* $|1/t^{\frac12}\varphi(1/t)|$ *exist.*

Proof. If $\varphi(p)\doteqdot f(t)$, then ($M_{(1)}$, p. 12)
$$\frac{p}{\pi^{\frac12}}\int_0^\infty e^{-\frac14 p^2x^2}\varphi(1/x^4)\,dx\doteqdot f(t^2).$$

Now, using (A), we have
$$\begin{aligned}\int_0^\infty U^{-1}\varphi(U)f(U^2)\,dU&=\frac{1}{\pi^{\frac12}}\int_0^\infty f(t)\left[\int_0^\infty e^{-\frac14 t^2x^2}\varphi(1/x^2)\,dx\right]dt \qquad (3.2)\\&=\frac{1}{\pi^{\frac12}}\int_0^\infty \varphi(1/x^2)\left[\int_0^\infty e^{-\frac14 t^2x^2}f(t)\,dt\right]dx\\&=\frac{1}{\pi^{\frac12}}\int_0^\infty x^{-2}\varphi(1/x^2)\psi(x^2)\,dx,\end{aligned}$$

where
$$\psi(p)\doteqdot\frac{f(2(t^{\frac12}))}{t^{\frac12}}.$$

The result follows after a little simplification.

Regarding the change of order of integrations in (3.2), we see that the t-integral and the x-integral are absolutely convergent due to the existence of the Laplace transforms of $|(1/(t^{\frac12}))f(2(t^{\frac12})|$ and $|1/(t^{\frac12})\varphi(1/t)|$, and the integral
$$\int_0^\infty x^{-2}\varphi(1/x^2)\psi(x^2)\,dx$$
is absolutely convergent due to the absolutely convergence of the integrals in (3.1). Hence the change of order is justified by the de la Vallée Poussin's Theorem (Bromwich, 1931, p. 504).

Example. Let ($M_{(1)}$, p. 19)
$$f(t)=\cos a\,t\doteqdot p^2/(p^2+a^2)=\varphi(p),\quad R(p)>0.$$

Then ($B_{(1)}$ p 158),
$$\psi(p)=(\pi^{\frac12})\,p^{\frac12}e^{-a^2/p},$$

Hence from (3.1), we get
$$\int_0^\infty\frac{Ve^{-aV}}{V^2+a}\,dV=\int_0^\infty\frac{\cos((a^{\frac12})U)}{U+a}\,dU$$

Now, evaluating the integral on the R.H.S. by a known integral ($B_{(1)}$, p. 8), we get
$$\int_0^\infty\frac{Ve^{-aV}}{V^2+a}\,dV=-\,Si\,(a^{3/2})\sin(a^{3/2})-Ci(a^{3/2})\cos(a^{3/2}),\ |\arg a|<\tfrac12\pi.$$

4. Theorem III. *If*

$$f(p) \doteq h(t),$$

$$h(p)p^{1-m}(1+p)^{-l} \doteq g(t),$$

and $$\Theta(p) \doteq F(t), \quad \textit{then}$$

$$\int_0^\infty h(V)\Theta(V)\frac{dV}{V} = \int_0^\infty g(s)\psi(s)ds \tag{4.1}$$

where

$$\psi(s) = \Gamma(1+m)\int_0^\infty F(U)\, e^{\frac{1}{2}(U+s)}(U+s)^{-\frac{1}{2}(l+m+2)}W_{\frac{1}{2}(l-m),\,\frac{1}{2}(l+m+1)}(U+s)dU \tag{4.2}$$

provided the integrals in (4.1) and (4.2) converge absolutely, the Laplace transforms of $|h(t)|$ *and* $|g(t)|$ *exist and*

$$\int_0^\infty g(s)e^{\frac{1}{2}(U+s)}(V+s)^{-\frac{1}{2}(l+m+2)}W_{\frac{1}{2}(l-m),\,\frac{1}{2}(l+m+1)}(U+s)\,ds$$

is absolutely convergent.

Proof. If $$f(p) \doteq h(t),$$

$$h(p)p^{1-m}(1+p)^{-l} \doteq g(t), \qquad \text{then (Shastri, 1945, p. 4)}$$

$$f(p) = p\Gamma(1+m)\int_0^\infty e^{\frac{1}{2}(p+s)}(p+s)^{-\frac{1}{2}(l+m+2)}g(s)W_{\frac{1}{2}(l-m),\,\frac{1}{2}(l+m+1)}(p+s)ds$$

Applying (A), we have

$$\int_0^\infty h(V)\Theta(V)\frac{dV}{V} = \int_0^\infty F(U)\Gamma(1+m)\Big[\int_0^\infty e^{\frac{1}{2}(U+s)}(U+s)^{-\frac{1}{2}(l+m+2)}g(s)$$

$$\times W_{\frac{1}{2}(l-m),\,\frac{1}{2}(l+m+1)}(U+s)ds\Big]dU \tag{4.3}$$

$$= \Gamma(1+m)\int_0^\infty g(s)\Big[\int_0^\infty F(U)e^{\frac{1}{2}(U+s)}(U+s)^{-\frac{1}{2}(l+m+2)}$$

$$\times W_{\frac{1}{2}(l-m),\,\frac{1}{2}(l+m+1)}(U+s)dU\Big]ds$$

$$= \int_0^\infty g(s)\psi(s)ds,$$

where

$$\psi(s) = \Gamma(1+m)\int_0^\infty F(U)e^{\frac{1}{2}(U+s)}(U+s)^{-\frac{1}{2}(l+m+2)}W_{\frac{1}{2}(l-m),\,\frac{1}{2}(l+m+1)}(U+s)dU.$$

The change of order of integrations in (4.3) can be justified in the same manner as in Theorem II.

Particular Cases:

(1) If $m = 1$, $l = 0$, then the Theorem III reduces to:

If $$f(p) \doteqdot h(t),$$
$$h(p) \doteqdot g(t),$$
and $$\theta(p) \doteqdot F(t), \quad \text{then}$$
$$\int_0^\infty h(V, \Theta V)\frac{dV}{V} = \int_0^\infty \psi(s) g(s) ds,$$
where
$$\psi(s) = \int_0^\infty \frac{F(V)}{(V+s)^2}\, dV$$

provided the integrals converge absolutely, the Laplace transforms of $|h(t)|$ and $|g(t)|$ exist, and $\int_0^\infty \frac{g(t)dt}{(V+t)^2}$ is absolutely convergent.

(2) Let in Theorem III, $m = 0$, and $h(t) = \frac{1}{(1+t)^\nu}$, where ν is a positive integer.

Then ($M_{(1)}$, p. 16)
$$g(t) = \frac{e^{-t} t^{\nu+l-1}}{\Gamma(\nu+l)}, \qquad \mathbf{R}(\nu+l) > 0$$

Hence from (4.1), we have
$$\frac{1}{\Gamma(\nu+l)} \int_0^\infty e^{-s} s^{\nu+l-1} \psi(s) ds = \int_0^\infty \frac{\Theta(V)}{V(1+V)^\nu}\, dV \tag{4.4}$$
where
$$\psi(s) = \int_0^\infty F(U) e^{\frac{1}{2}(U+s)} (U+s)^{-\frac{1}{2}(l+2)} W_{\frac{1}{2}l, \frac{1}{2}(l+1)}(U+s) dU \tag{4.5}$$

provided $\mathbf{R}(\nu+l) > 0$, ν is a positive integer end the integrals converge absolutely.

Example.

Let ($M_{(1)}$, p. 10)
$$F(t) = \frac{e^{-t}}{t^{\frac{1}{2}}} \doteqdot \frac{\pi^{\frac{1}{2}} p}{(p+1)^{\frac{1}{2}}} = \Theta(p), \quad \mathbf{R}(p+1) > 0$$

Hence from (4.4) and (4.5), we get
$$\int_0^\infty e^{-s} s^{\nu+l-1} \psi(s) ds = \frac{2(\pi^{\frac{1}{2}})\Gamma(\nu+l)}{(2\nu-1)} \tag{4.6}$$
where
$$\psi(s) = \int_0^\infty U^{-\frac{1}{2}} e^{\frac{1}{2}(s-U)} (U+s)^{-\frac{1}{2}(l+2)} W_{\frac{1}{2}l, \frac{1}{2}(l+1)}(U+s)\, dU$$

$R(\nu+l) > 0$ and ν is a positive integer.

5 **Theorem IV:** *If $f(x)$ is Hankel transform of $g(x)$ of order ν and $\psi(p) \doteqdot f(y)F(y)$, then*

$$\psi(p) = \int_0^\infty g(x)\varphi(p, x)dx \tag{5.1}$$

where $\varphi(p, x) \doteqdot (xy)^{\frac{1}{2}}J_\nu(xy)F(y)$,

provided the integrals involved are absolutely convergent.

Proof. We have

$$f(y) = \int_0^\infty (xy)^{\frac{1}{2}}J_\nu(xy)g(x)dx \tag{5.2}$$

Multiplying both sides of the equation (5.2) by $e^{-py}F(y)$ and integrating w.r.t 'y' between the limits 0 to ∞, we get

$$p\int_0^\infty e^{-py}f(y)F(y)dy = p\int_0^\infty e^{-py}F(y)\left[\int_0^\infty (xy)^{\frac{1}{2}}J_\nu(xy)g(x)dx\right]dy \tag{5.3}$$

Now, interchanging the order of integrations, we get

$$\psi(p) = \int_0^\infty g(x)\left[p\int_0^\infty e^{-py}(xy)^{\frac{1}{2}}J_\nu(xy)F(y)dy\right]dx$$

$$= \int_0^\infty g(x)\varphi(p, x)dx,$$

where $\varphi(p, x) \doteqdot (xy)^{\frac{1}{2}}J_\nu(xy)F(y)$.

The change of order of integrations in (5.3) can be justified as in Theorem II.

Example. Let (Bateman Project, Vol. 2, 1954, p. 82) *

$$g(x) = x^{-1}e^{-\frac{1}{2}x^2}M_{\frac{1}{4}-\frac{1}{4},\frac{1}{2}\nu+\frac{1}{4}}(x^2), \qquad \mathbf{R}(\nu) > -\tfrac{1}{2}$$

Then

$$f(x) = \frac{(2\nu+1)}{2^\nu}x^{-\frac{1}{2}}\operatorname{erfc}(\tfrac{1}{2}x).$$

Further, let $F(x) = x^{-(\nu-\frac{1}{2})}$

Then ($M_{(2)}$, p. 87)

$$\psi(p) = \frac{2\nu+1}{2^\nu}(1-e^{p^2}\operatorname{erfc} p)$$

Also ($B_{(1)}$), p. 182)

$$\varphi(p, x) = \frac{(x^{\frac{1}{2}})p}{(p^2+x^2)^{(2-\nu)/2}}P^{-\nu}_{1-\nu}\left(\frac{p}{(p^2+x^2)^{\frac{1}{2}}}\right) \tag{5.4}$$

$\mathbf{R}(p) > 0$.

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* This book will be denoted by $B_{(2)}$ in this paper.

Hence from Theorem IV, we get

$$\int_0^\infty x^{-1}e^{-\frac14x^2}M_{\frac14\nu\mp\frac14,\,\frac14\nu+\frac14}(x^2)\varphi(p,\,x)\,dx$$

$$=(2\nu\pm1)/2^\nu\;(1-e^{p^2}\operatorname{erfc}p),$$

$\mathbf{R}(p)>0$, $\mathbf{R}(\nu)>-\frac12$ and $\varphi(p,\,x)$ is given by (5.4).

6. Theorem V. *If*

$$\varphi(p)\doteqdot f(t),$$

$$h(p)\doteqdot f((t)^{\frac12})/(t^{\frac12}).$$

and

$$\chi(p)\doteqdot F(t),$$

then, provided that the integrals involved converge absolutely,

$$\frac{2}{\pi^{\frac12}}\int_0^\infty t^{-\frac12}\chi(t)h(1/4t)\,dt=\int_0^\infty U^{-1}F(U)\varphi(U^{\frac12})\,dU \tag{6.1}$$

Proof. We know ($M_{(2)}$, p. 19,) if

$$\varphi(p)\doteqdot f(t),\quad h(p)\doteqdot\frac{f(t^{\frac12})}{t^{\frac12}},$$

then

$$\varphi(p^{\frac12})\doteqdot 2(t/\pi)^{\frac12}h(1/4t)$$

we get the required result by using (A).

Examples. (1) Let ($B_{(1)}$, p, 182)

$$F(t)=t^\mu J_\nu(at)\doteqdot\frac{\Gamma(\mu+\nu+1)\,p}{(p^2+a^2)^{\frac12(\mu\pm1)}}P_\mu^{-\nu}\left(\frac{p}{((p^2+a^2)^{\frac12})}\right)$$

$$=\chi(p).$$

$\mathbf{R}(\mu+\nu+1)>0$, $\mathbf{R}(p)>|\,\mathbf{Im}\,a\,|$.

Inserting these in (6.1) with $f(t)=\cos t$, we get

$$\int_0^\infty\frac{e^{-t}}{(t^2+a^2)^{\frac12(\mu+1)}}P_\mu^{-\nu}\left(\frac{t}{(t^2+a^2)^{\frac12}}\right)dt=\frac{1}{\Gamma(\mu+\nu+1)}\int_0^\infty\frac{U^\mu J_\nu(aU)}{U+1}\,dU$$

Now, evaluating the integral on the R. H. S. with the help of a known integral (Watson, 1952, p. 436)†, we get

$$\int_0^\infty\frac{e^{-t}}{(t^2+a^2)^{\frac12(\mu+1)}}P_\mu^{-\nu}\left(\frac{t}{(t^2+a^2)^{\frac12}}\right)dt=\frac{\pi}{\Gamma(\mu+\nu+1)\sin(\mu+\nu+1)\pi}$$

$$\times\left[J_\nu(a)-\sum_{m=0}^\infty\frac{(\frac12a)^{m-\mu}\sin\frac12(\mu+\nu+1-m)\pi}{\Gamma_*(\frac12m-\frac12\mu+1\pm\frac12\nu)}\right] \tag{6.2}$$

$\mathbf{R}(\mu+\nu+1)>0$, $\mathbf{R}(\mu)<\frac12$, $a>0$.

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† We shall quote this work as B. F. in this paper.

* The symbol $\Gamma_*(\alpha\pm\beta)$ denotes $\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)$.

In particular, if we put $\mu = \nu$, then (B. F. p. 486),

$$\int_0^\infty \frac{e^{-t}}{(t^2+a^2)^{\frac{1}{2}(\nu+1)}} P_\nu^{-\nu}\left(\frac{t}{(t^2+a^2)^{\frac{1}{2}}}\right)dt = \frac{\pi}{2\Gamma(2\nu+1)\cos\nu\pi}[H_{-\nu}(a) - Y_{-\nu}(a) \tag{6.3}$$

provided that $a > 0$ and $-\frac{1}{2} \leq \mathbf{R}(\nu) < \frac{1}{2}$

(2) Let $(B_{(1)}$, p. 212)

$$F(t) = t^{\gamma-\frac{1}{2}}\,{}_2F_1[\alpha, \beta; \gamma; -t]$$

$$\doteqdot \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)p^{\gamma-\frac{1}{2}}} E(\alpha, \beta, \gamma+\tfrac{1}{2} : \gamma : p) = X(p),$$

$\mathbf{R}(\gamma+\frac{1}{2}) > 0$ and $\mathbf{R}(p) > 0$.

Then (6.1) with $f(t) = J_0(at)$, reduces to

$$\frac{\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)}\int_0^\infty t^{-\gamma-\frac{1}{2}} e^{-\frac{1}{2}a^2t} I_0(\tfrac{1}{2}a^2t) E(\alpha, \beta, \gamma+\tfrac{1}{2}; \gamma; t)\,dt$$

$$= \int_0^\infty \frac{U^{\gamma-1}}{(U+a^2)^{\frac{1}{2}}}\,{}_2F_1[\alpha, \beta; \gamma; -U]\,dU.$$

Now, evaluating the R.H.S. by a known integral $(B_{(2)}$, p. 400), we get

$$\int_0^\infty t^{-\gamma-\frac{1}{2}} e^{-\frac{1}{2}a^2t} I_0(\tfrac{1}{2}a^2t)\, E(\alpha, \beta, \gamma+\tfrac{1}{2}; \gamma; t)\,dt$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha-\gamma+\frac{1}{2})\Gamma(\beta-\gamma+\frac{1}{2})}{(\pi^{\frac{1}{2}})\,\Gamma(\alpha+\beta-\gamma+\frac{1}{2})}\,{}_2F_1\left[\begin{matrix}\alpha-\gamma+\frac{1}{2},\ \beta-\gamma+\frac{1}{2} \\ \alpha+\beta-\gamma+\frac{1}{2}\end{matrix}\ ;\ 1-a^2\right] \tag{6.4}$$

provided $\mathbf{R}(\gamma) > 0$, $\mathbf{R}(\alpha-\gamma+\frac{1}{2}) > 0$, $\mathbf{R}(\beta-\gamma+\frac{1}{2}) > 0$ and $|\arg a^2| < \pi$.

Theorem VI. *If* $\varphi(p) \doteqdot f(t)$,

$h(p) \doteqdot 1/(t^{\frac{1}{2}})\varphi(1/t)$,

and $X(p) \doteqdot F(t)$, *then*

$$\frac{2}{\pi^{\frac{1}{2}}}\int_0^\infty t^{-\gamma}F(t)h(t^2/4)\,dt = \int_0^\infty f(U^2)X(U)\,\frac{dU}{U} \tag{7.1}$$

provided the integrals involved are absolutely convergent.

Proof. We know $(M_{(2)}$, p. 19) that, if

$\varphi(p) \doteqdot f(t)$, $h(p) \doteqdot 1/t^{\frac{1}{2}}\varphi(1/t)$, then

$2/p(\pi^{\frac{1}{2}})h(p^2/4) \doteqdot f(t^2)$

Also $X(p) \doteqdot F(t)$.

Substituting these in (A), we get the desired result.

8. **Theorem VII.** *If*

$$\varphi(p) \doteqdot f(t),$$

$$p^{\frac{1}{2}}\varphi(p^{\frac{1}{2}}) \doteqdot 1/(\pi t)^{\frac{1}{2}}h(1/t),$$

and

$$\chi(p) \doteqdot F(t), \textit{ then}$$

$$\int_0^\infty h(t)F(t)\,dt/t = \int_0^\infty \chi(U)f(2(U^{\frac{1}{2}}))dU/U \tag{8.1}$$

provided the integrals involved are absolutely convergent.

Proof. We know (Sarkar, 1955, p. 81) that, if

$$\varphi(p) \doteqdot f(t),$$

$$p^{\frac{1}{2}}\varphi(p^{\frac{1}{2}}) \doteqdot 1/(\pi t)^{\frac{1}{2}}h(1/t), \text{ then}$$

$$h(p) \doteqdot f(2(t^{\frac{1}{2}})).$$

Also

$$\chi(p) \doteqdot F(t).$$

Now the theorem follows by substituting these values in (*A*).

Example. Let ($M_{(1)}$, p. 51)

$$F(t) = \frac{a^{2\mu}}{t^{\mu}}e^{-a^2/8t}W_{\mu,\nu}(a^2/4t)$$

$$\doteqdot \frac{a^{2\mu+1}}{p^{\frac{1}{2}(2\mu-1)}}K_{2\nu}(a(p^{\frac{1}{2}})) = \chi(p), \qquad \mathbf{R}(p) \geq 0$$

Therefore, from (8.1) with $f(t) = t^{n-1}e^{-t^2}$, we get

$$\Gamma(\tfrac{1}{2}n+\tfrac{1}{2})\int_0^\infty t^{-\mu}(t+4)^{-\frac{1}{2}(n+1)}e^{-a^2/8t}W_{\mu,\nu}(a^2/4t)dt$$

$$= a\int_0^\infty x^{\frac{1}{2}n-\mu-1}e^{-4x}K_{2\nu}(a(x^{\frac{1}{2}}))dx.$$

Now, evaluating the integral on the R.H.S. by a known integral ($B_{(1)}$, p. 199), we get

$$\int_0^\infty t^{-\mu}(t+4)^{-\frac{1}{2}(n+1)}e^{-a^2/8t}W_{\mu,\nu}(a^2/4t)dt$$

$$= \frac{\Gamma_*(\frac{1}{2}n-\mu\pm\nu)\;e^{a^2/32}}{2^{n-2\mu-1}\Gamma(\frac{1}{2}n+\frac{1}{2})}W_{-(\frac{1}{2}n-\mu-\frac{1}{2}),\nu}(a^2/16) \tag{8.2}$$

provided $\mathbf{R}(\frac{1}{2}n-\mu\pm\nu)>0$ and $\mathbf{R}(a)\geq 0$.

Theorem VIII. *If*

$$\varphi(p) \doteqdot f(t)$$

$$h(p) \doteqdot f(2(t^{\frac{1}{2}})),$$

and $\chi(p) \doteqdot F(t)$, *then, provided the integrals involved converge absolutely.*

$$\int_0^\infty t^{-\frac{1}{2}}\varphi(t^{\frac{1}{2}})F(t)dt = \frac{1}{\pi^{\frac{1}{2}}}\int_0^\infty x^{-3/2}h(1/x)\chi(x)dx \tag{9.1}$$

Proof. The proof follows by using another result due to Sarkar and proceeding as in Theorem VII.

10. **Theorem IX.** *If*

$$f(p) \doteqdot h(t),$$

and

$$\psi(p) \doteqdot e^{\frac{1}{4}t^2}D_{-\nu}(t)h(t),$$

then

$$\frac{p}{\Gamma(\nu)}\int_0^\infty x^{\nu-1}e^{-\frac{1}{2}x^2}\frac{f(p+x)}{(p+x)}dx = \psi(p) \tag{10.1}$$

provided the integrals converge absolutely, $\mathbf{R}(\nu) > 0$ *and the Laplace transform of* $|h(t)|$ *exists.*

Proof. We have
$$\frac{f(p+x)}{(p+x)} = \int_0^\infty e^{-(p+x)t}h(t)dt \tag{10.2}$$

Multiplying both sides of (10.2) by $x^{\nu-1}e^{-\frac{1}{2}x^2}$ and integrating w.r.t. x between the limits 0 to ∞, we get

$$\int_0^\infty x^{\nu-1}e^{-\frac{1}{2}x^2}\frac{f(p+x)}{(p+x)}dx = \int_0^\infty x^{\nu-1}e^{-\frac{1}{2}x^2}\left[\int_0^\infty e^{-(p+x)t}h(t)dt\right]dx \tag{10.3}$$

Changing the order of integrations and using the result (Whittaker, 1903).

$$\text{(B)} \qquad D_{-n}(z) = \frac{1}{\Gamma(n)}e^{-\frac{1}{4}z^2}\int_0^\infty e^{-tz-\frac{1}{2}t^2}\,t^{n-1}dt, \qquad \mathbf{R}(n) > 0,$$

We get

$$\frac{p}{\Gamma(\nu)}\int_0^\infty x^{\nu-1}e^{-\frac{1}{2}x^2}\frac{f(p+x)}{(p+x}dx = \psi(p),$$

where

$$\psi(p) \doteqdot e^{\frac{1}{4}t^2}D_{-\nu}(t)h(t).$$

The change of the order of integrations in (10 3) can be justified as in Theorem II.

11. **Theorem X.** *If* $\varphi(p) \doteqdot f(x)$, *and* $p^\lambda\varphi(p)$ *is self-reciprocal in the Hankel transform of order* ν, *then*

$$\int_0^\infty x^{-\lambda-\nu-5/2}\,{}_2F_1\left[\frac{2\lambda+2\nu+5}{4},\frac{2\lambda+2\nu+7}{4};\nu+1;-\frac{t^2}{x^2}\right]f(x)dx = \frac{2^{\lambda+3/2}\Gamma(\nu+1)t^{-\nu-\frac{1}{2}}}{\Gamma(\lambda+\nu+\frac{5}{2})}\varphi(t) \tag{11.1}$$

provided the integrals converge absolutely, $\mathbf{R}(\lambda+\nu+\frac{5}{2})>0$ *and the Laplace transform of* $|f(x)|$ *exists.*

Proof. We have

$$\varphi(y) = y\int_0^\infty e^{-xy} f(x)\,dx \tag{11.2}$$

Multiplying both sides of (11.2) by $y^\lambda \sqrt{(yt)}J_\nu(yt)$ and integrating w.r.t. y between the limits zero and infinity, we get

$$\varphi(t) = \int_0^\infty \sqrt{(yt)}J_\nu(yt)y^\lambda\Big[y\int_0^\infty e^{-xy}f(x)dx\Big]dy \tag{11.3}$$

Changing the order of integration, we obtain

$$\varphi(t) = \int_0^\infty \frac{f(x)}{x}\Big[x\int_0^\infty e^{-xy}y^{\lambda+1}\sqrt{(yt)}J_\nu(yt)dy\Big]dx.$$

Evaluating the inner integral ($M_{(1)}$, p. 28) we get the required result.

The change of order of integrations in (11.3) can be justified by de la Vallée Poussin's Theorem.

12. **Theorem XI.** *If*

$$\psi(p) \doteqdot \varphi(t),$$

$$\varphi(p) \doteqdot f_1(t),$$

$$\sqrt{p}\, f_1(p) \doteqdot f_2(t),$$

and

$$f_2(p) \doteqdot g(t),$$

then

$$\psi(p) = \frac{\pi p}{2}\int_0^\infty \frac{((pt)^{\frac14}-1)^3}{(pt-1)^3}\,\frac{g(t)}{t^{\frac14}}\,dt \tag{12.1}$$

provided that the integrals are absolutely convergent, the Laplace transforms of $|t^2 f_2(t^2/4)|$ *and* $|g(t)|$ *exist and the integral* $\int_0^\infty e^{-t/x}f_2(t)dt$ *is absolutely convergent.*

Proof. If $\varphi(p) \doteqdot f_1(t)$, $p^{\frac12}f_1(1/p) \doteqdot f_2(t)$,

and $f_2(p) \doteqdot g(t)$, then under the conditions stated, (Bose, 1953, p. 5) we have

$$\varphi(p^2) = 3\pi^{\frac12}p\int_0^\infty t^{-2}e^{p^2/2t}D_{-4}(p(2/t)^{\frac12})g(t)dt \tag{12.2}$$

Also

$$\psi(p) = p\int_0^\infty e^{-px}\varphi(x)dx \tag{12.3}$$

Substituting the value of $\varphi(x)$ from (12.2) in (12.3) and changing the order of integrations, which can be justified as in Theorem II, we get

$$\psi(p) = 8\pi^{\frac{1}{2}}p\int_0^\infty t^{-2}g(t)\left[\int_0^\infty x^{\frac{1}{2}}e^{(1/2t-p)x}D_{-4}\left(\left(\frac{2x}{t}\right)^{\frac{1}{2}}\right)dx\right]dt.$$

Now, evaluating the inner integral by the known integral (Goldstein, *loc. cit.*, p. 118)

$$\text{(C)} \qquad \int_0^\infty y^{\nu}e^{(\frac{1}{4}-\alpha)y^2}D_{-(\nu+2)}(y)dy = \left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{[(2\alpha)^{\frac{1}{2}}-1]^{\nu+1}}{(\nu+1)(2\alpha-1)^{\nu+1}}, \left(\nu > -1, \ |\,\text{aeg}\,\alpha| < \frac{\pi}{3}\right)$$

We have $$\psi(p) = \frac{\pi p}{2}\int_0^\infty \frac{((pt)^{\frac{1}{2}}-1)^3}{(pt-1)^3}\,\frac{g(t)}{t^{\frac{1}{2}}}\,dt.$$

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ON THE EFFECT OF PRESCRIBED SHEARING STRESS ON THE PLANE SURFACE OF AN ELASTIC SLAB WITH VARYING MODULUS OF RIGIDITY

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Introduction

The problem of a semi-infinite elastic isotropic medium with torsional displacement prescribed on the surface has been discussed by Reissner and Sagoci (1944). Sneddon (1947) has obtained the same result by the use of Hankel transform. The object of this paper is to obtain the stress distribution when the shearing stress is prescribed on the plane surface of an elastic slab resting on a rigid foundation, the modulus of rigidity of the medium varying exponentially with depth.

Method of Solution

1. Let the origin be taken on the plane boundary $z = 0$, the other being $z = b$, the axis of z being drawn into the body at right angles to this plane.

Equations of equilibrium in the absence of body forces are

$$\left.\begin{aligned}
&\frac{\partial}{\partial r}(\widehat{rr}) + \frac{1}{r}\frac{\partial}{\partial \theta}(\widehat{r\theta}) + \frac{1}{r}(\widehat{rr} - \widehat{\theta\theta}) = 0\\
&\frac{\partial}{\partial r}(\widehat{r\theta}) + \frac{1}{r}\frac{\partial}{\partial \theta}(\widehat{\theta\theta}) + \frac{\partial}{\partial z}(\widehat{\theta z}) + \frac{2}{r}\widehat{r\theta} = 0\\
&\frac{\partial}{\partial r}(\widehat{rz}) + \frac{1}{r}\frac{\partial}{\partial \theta}(\widehat{\theta z}) + \frac{\partial}{\partial z}(\widehat{zz}) + \frac{1}{r}\widehat{rz} = 0
\end{aligned}\right\} \tag{1.1}$$

For solving the problem under consideration, let us assume

$$u_r = 0, \; u_\theta = v(r, z), \; u_z = 0 \tag{1.2}$$

where u_r, u_θ, u_z are components of displacement.

Then

$$\left.\begin{aligned}
&e_{rr} = e_{\theta\theta} = e_{zz} = e_{rz} = 0\\
&\widehat{rr} = \widehat{\theta\theta} = \widehat{zz} = \widehat{rz} = 0\\
&\widehat{r\theta} = \mu\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)\\
&\widehat{\theta z} = \mu\frac{\partial v}{\partial z}.
\end{aligned}\right\} \tag{1.3}$$

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Two of the equations of equilibrium (1.1) are identically satisfied and the third reduces to

$$\frac{\partial}{\partial r}\left[\mu\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)\right]+\frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right)+\frac{2}{r}\mu\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)=0 \tag{1.4}$$

If

$$\mu=\mu_0 e^{kz} \tag{1.5}$$

where μ_0, k are constants, the equation (1.4) becomes

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}+\frac{\partial^2 v}{\partial z^2}+k\frac{\partial v}{\partial z}=0 \tag{1.6}$$

Let

$$v=RZ$$

where R is a function of r only and Z, a function of z alone. Hence we shall have from the equation (1.6):

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}-\frac{R}{r^2}+\alpha^2 R=0 \tag{1.7}$$

$$\frac{d^2Z}{dz^2}+k\frac{dZ}{dz}-\alpha^2 Z=0 \tag{1.8}$$

The solution of equation (1.8) is obviously given by

$$Z=Ae^{\gamma_1 z}+Be^{\gamma_2 z}$$

where

$$\gamma_1=\tfrac{1}{2}((k^2+4\alpha^2)^{\frac{1}{2}}-k),\quad \gamma_2=-\tfrac{1}{2}((k^2+4\alpha^2)^{\frac{1}{2}}+k) \tag{1.9}$$

A solution of equation (1.7) is

$$R=CJ_1(\alpha r)$$

where J_1 is Bessel's function of the first kind and first order.

Hence we can put

$$v=\int_0^\infty \alpha[A(\alpha)e^{\gamma_1 z}+B(\alpha)e^{\gamma_2 z}]J_1(\alpha r)d\alpha \tag{1.10}$$

in which $A(\alpha)$, $B(\alpha)$ are arbitrary functions of α only. We can now write down the non-vanishing components of stress from relations (1.3).:

$$\left.\begin{aligned}\widehat{r\theta}&=\int_0^\infty \alpha\mu_0 e^{kz}[A(\alpha)e^{\gamma_1 z}+B(\alpha)e^{\gamma_2 z}]\,[J'_1(\alpha r)-(1/r)J_1(\alpha r)]\,d\alpha\\ \widehat{\theta z}&=\int_0^\infty \alpha\mu_0 e^{kz}[\gamma_1 A(\alpha)e^{\gamma_1 z}+\gamma_2 B(\alpha)e^{\gamma_2 z}]J_1(\alpha r)d\alpha\end{aligned}\right\} \tag{1.11}$$

2. Since the elastic slab is supposed to rest on a rigid foundation we shall have

$$v=0 \text{ when } z=b,$$

the boundary $z = b$ being rigidly fixed. So we have from the relation (1.10)

$$A(\alpha)e^{\gamma_1 b}+B(\alpha)e^{\gamma_2 b}=0 \tag{2.1}$$

Also we shall have

$$(\widehat{\theta z})_{z=0}=f(r) \tag{2.2}$$

where $f(r)$ is a given function.

From Fourier-Bessel integral we can write

$$f(r)=\int_0^\infty \alpha J_1(\alpha r)\,d\alpha\int_0^\infty yf(y)J_1(\alpha y)dy. \tag{2.3}$$

Hence we get from the relations (1.11) and (2.3) and the condition (2.2)

$$\gamma_1 A(\alpha)+\gamma_2 B(\alpha)=\frac{1}{\mu_0}\int_0^\infty yf(y)J_1(\alpha y\, dy. \tag{2.4}$$

The integral (2.4) cannot be evaluated for all values of $f(y)$. We therefore assume

$$\begin{aligned} f(r) &= Qr, \text{ for } 0\leq r\leq a \\ &= 0, \text{ for } a<r \end{aligned} \tag{2.5}$$

where Q is a constant. Then we have

$$\begin{aligned}\gamma_1 A(\alpha)+\gamma_2 B(\alpha) &=\frac{Q}{\mu_0}\int_0^a y^2J_1(\alpha y)dy \\ &=\frac{Qa^2}{\mu_0\alpha}J_2(\alpha a).\end{aligned} \tag{2.6}$$

The equations (2.1) and (2.6) will be satisfied if we write

$$\begin{aligned} A(\alpha) &=\frac{Qa^2}{\mu_0\alpha P}e^{\gamma_2 b}J_2(\alpha a) \\ B(\alpha) &=-\frac{Qa^2}{\mu_0\alpha P}e^{\gamma_1 b}J_2(\alpha a) \end{aligned} \tag{2.7}$$

where

$$P=\gamma_1 e^{\gamma_2 b}-\gamma_2 e^{\gamma_1 b}. \tag{2.8}$$

3. The displacement component on the surface $z = 0$ is given by

$$\begin{aligned}(v)_{z=0} &=\int_0^\infty \alpha[A(\alpha)+B(\alpha)]J_1(\alpha r)d\alpha \\ &=\frac{Qa^2}{\mu_0}\int_0^\infty\frac{1}{P}(e^{\gamma_2 b}-e^{\gamma_1 b})J_2(\alpha a)J_1(\alpha r)d\alpha\end{aligned} \tag{3.1}$$

To find how the displacement component on the surface $z = 0$ varies for different values of r we calculate the integral (3.1) numerically and suppose for example:

$$a=b=1,\quad k=1$$

The results of calculation for $(v)_{z=0}\mu_0/Q$ for different values of r are given in Table 1.

TABLE 1.

$r = 0$	.25	.50	.75	1.00
$(v)_{z=0}\mu_0/Q = 0$	—.123	—.196	—.192	—.140

SHEARING STRESSES DISTRIBUTED UNIFORMLY OVER A CIRCULAR AREA.

4. If we suppose that shearing stresses are distributed uniformly over a circular area on the upper surface $z = 0$, we can assume

$$f(r) = S, \text{ for } 0 \leq r \leq a$$
$$= 0, \text{ for } a < r$$

where S is a constant and the resultant torque

$$M = \int_0^a S(2\pi r)rdr = \tfrac{2}{3}S\pi a^3,$$

so that
$$S = 3M/2\pi a^3$$

Hence we get from the relations (1.11) and (2.3) and the condition (2.2)

$$\gamma_1 A(\alpha) + \gamma_2 B(\alpha) = \frac{3M}{2\pi\mu_0 a^3}\int_0^a yJ_1(\alpha y)dy$$
$$= \frac{3M}{2\pi\mu_0 a^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,(n+1)!}\left(\frac{\alpha}{2}\right)^{2n+1}\frac{a^{2n+3}}{2n+3} \tag{4.2}$$

By solving the equations (2.1) and (4.2) we get the values of $A(\alpha)$ and $B(\alpha)$ as

$$A(\alpha) = \frac{3Me^{\gamma_1 b}}{2\pi\mu_0 a^3 P}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,(n+1)!}\left(\frac{\alpha}{2}\right)^{2n+1}\frac{a^{2n+3}}{2n+3}$$
$$B(\alpha) = -\frac{3Me^{\gamma_1 b}}{2\pi\mu_0 a^3 P}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,(n+1)!}\left(\frac{\alpha}{2}\right)^{2n+1}\frac{a^{2n+3}}{2n+3} \tag{4.3}$$

where P is given by the relation (2.8)

Thus the stresses and displacement are given by the relations (1 10) and (1.11) with the values of the constants as in (4.3).

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ON AXIALLY SYMMETRIC SUPERPOSABLE FLOWS

BY

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1. Introduction. Axially symmetric superposable flows have been considered by Bhatnagar and Varma (1957), Lakshman Rao (1957), Prem Prakash (1955) and Ram Ballabh (1955).

Ram Ballabh (1955) showed that two non-viscous axially symmetric flows $(\vec{q}_1, \vec{\zeta}_1, p_1, \Omega_1)$ and $(\vec{q}_2, \vec{\zeta}_2, p_2, \Omega_2)$ are superposable in the following cases:

(*a*) each flow is irrotational;

(*b*) each has constant vorticity;

(*c*) Vorticity of each is constant along the stream-lines of the other, and either the flows are purely axial or the ratio of their vorticities is equal and opposite in sign to the ratio of their respective radial components of velocities, the axial components of the velocities not being zero in both the flows.

Bhatnagar and Varma (1957) showed that in the case of two viscous axially symmetric flows, the condition of superposability is satisfied in case (*a*) above and in cases:

(*d*) vorticity of each is proportional to r, the distance from the axis;

(*e*) ζ/r for each flow is constant along the stream lines of other flow.

Bhatnagar and Varma (1957) also showed that the condition of superposability is the same for steady and non-steady flows; of course, the conditions of integrability of the equations of motions are different in the two cases. A similar result about viscosity, though not stated explicitly there follows easily from their treatment. Bhatnagar and Lakshman Rao (1957) have proved a similar result about the effect of cross viscosity term in the case of superposability of two-dimensional non-Newtonian flows.

In the present paper we have shown that case (*b*) is not applicable to viscous flows. We have also shown that while in cases (*a*), (*b*), (*d*), the conditions of integrability of the equations of motions are also satisfied, this is not always true in the other two cases. We have also extended case (c) to cover a whole class of similar flows in which the condition of superposability will be satisfied. In particular we have shown that if two steady non-viscous axially symmetric flows are superposable, then their stream-lines have to coincide.

Prem Prakash (1955) examined some particular solutions for non-steady self-superposable flows. In his most important solution, he integrated the resulting differential equation in series. We show in section 4 that the differential equation is effectively equivalent to Bessel's differential equation of order unity and we are thus able to consider a more general solution than his

Finally, we also examine the problem of decay of vorticity for self-superposable axially symmetric flows when the velocities have toroidal as well as poloidal components.

2. Cases where superposability condition is satisfied. Let $u_1, v_1; u_2, v_2$ be the axial and radial components of velocities of the two flows and let ψ_1, ψ_2 be the corresponding Stokes' stream functions, so that

$$u_i = -\frac{1}{r}\frac{\partial\psi_i}{\partial r}, \quad v_i = \frac{1}{r}\frac{\partial\psi_i}{\partial z} \quad (i = 1, 2) \tag{1}$$

$$\zeta_i = r^{-1}E^2\psi_i \quad (i = 1, 2) \tag{2}$$

where

$$E^2 \equiv \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{3}$$

From (Bhatnagar and Varma, 1957), the conditions of integrability for steady flows are :

$$r\frac{\partial(\psi_i, r^{-2}E^2\psi_i)}{\partial(z, r)} = \nu E^4\psi_i, \quad (i = 1, 2) \tag{4}$$

and the condition of superposability is

$$\frac{\partial(\psi_1, r^{-2}E^2\psi_2)}{\partial(z, r)} + \frac{\partial(\psi_2, r^{-2}E^2\psi_1)}{\partial(z, r)} = 0 \tag{5}$$

Using (2) and (3), (4) can be written as

$$r\frac{\partial(\psi_i, \zeta_i/r)}{\partial(z, r)} = \nu E^4(\psi_i) = \nu E^2(r\zeta_i) \tag{6a}$$

or

$$\frac{\partial(\psi_i, \zeta_i)}{\partial(z, r)} - \frac{\zeta_i}{r}\frac{\partial\psi_i}{\partial z} = \nu E^2(r\zeta_i) \tag{6b}$$

Also the condition of superposability (5) can be written as

$$\frac{\partial(\psi_1, \zeta_2/r)}{\partial(z, r)} + \frac{\partial(\psi_2, \zeta_1/r)}{\partial(z, r)} = 0 \tag{7a}$$

or

$$\frac{\partial(\psi_1, \zeta_2)}{\partial(z, r)} + \frac{\partial(\psi_2, \zeta_1)}{\partial(z, r)} = \frac{1}{r}\left[\zeta_2\frac{\partial\psi_1}{\partial z} + \zeta_1\frac{\partial\psi_2}{\partial z}\right] \tag{7b}$$

or

$$\frac{\partial(\psi_1, \zeta_2)}{\partial(z, r)} + \frac{\partial(\psi_2, \zeta_1)}{\partial(z, r)} = v_1\zeta_2 + v_2\zeta_1 \tag{7c}$$

We now discuss the cases mentioned in section **1**.

Case (*a*). In this case $\zeta_1 = 0$, $\zeta_2 = 0$ and both (6) and (7) are satisfied.

Case (*d*). In this case ζ_1/r and ζ_2/r are constant and again both (6) and (7) are satisfied.

Case (*b*). If ζ_1 const. $= k_1 \neq 0$ and ζ_2 const. $= k_2 \neq 0$, then (6) will be satisfied if

$$v_1 = v_2 = \nu, \tag{8}$$

and (7) will be satisfied if

$$v_1 k_2 + v_2 k_1 = 0 \tag{9}$$

For both (8) and (9) to be satisfied, $v_1 = v_2 = 0$, $\nu = 0$ *i.e.*, for axially symmetric flows with constant non-zero vorticities to be superposable, the liquids have to be non-viscous and the flows purely axial. Thus this case discussed by Ram Ballabh (1955) cannot be extended to viscous flows.

Moreover for this case, from (1), ψ_1 and ψ_2 are functions of r only, and since

$$\zeta_i = \frac{\partial v_i}{\partial z} - \frac{\partial u_i}{\partial r},$$

we get

$$u_1 = c_1 - k_1 r, \quad u_2 = c_2 - k_2 r \tag{10}$$

where c_1, c_2 are constants, so that in this case the velocities are purely axial and change linearly with the distance from the axis.

Cases (*c*) *and* (*e*). Let

$$\zeta_1 = f(r) W_1, \; \zeta_2 = f(r) W_2 \tag{11}$$

then (7) becomes

$$f(r)\left\{\frac{\partial(\psi_1, W_2)}{\partial(z, r)} + \frac{\partial(\psi_2, W_1)}{\partial(z, r)}\right\} = \frac{f(r) - rf'(r)}{f(r)} \left[\zeta_1 v_2 + \zeta_2 v_1\right] \tag{12}$$

If W_1 is constant along the stream lines of the second flow and W_2 is constant along the stream lines of the first flow, then (12) will be satisfied if

either
$$f(r) = rf'(r) \text{ i.e. } f(r) = Ar \tag{13}$$

or
$$\zeta_1 v_2 + \zeta_2 v_1 = 0 \tag{14}$$

The first possibility implies that ζ/r for each flow is constant along the stream lines of the other flow. This corresponds to case (*e*). If (14) is satisfied, then the superposability condition will be satisfied not only when ζ for each flow is constant along the stream lines of the other flow—the statement of case (*c*)—but even when $\zeta/f(r)$ for each flow is constant along stream lines of the other flow. In this sense, this is a generalisation of case (*c*). We have thus established:

Theorem I. *If* $\zeta_1 v_2 + \zeta_2 v_1 \neq 0$, *then two flows which are such that for each flow* $\zeta/f(r)$ *is constant along the stream lines of the other flow will satisfy the condition of superposability if and only if*

$$f(r) = Ar.$$

Theorem II. *If* $\zeta_1 v_2+\zeta_2 v_1 = 0$, *the two flows which are such that for each flow* $\zeta/f(r)$ *is constant along the stream lines of the other will satisfy the condition of superposability for all differentiable functions* $f(r)$. *In particular, two purely axial flows will satisfy the superposability condition if* $\zeta/f(r)$ *for each flow is constant along the stream lines of the other.*

3. Self-superposable flows. For two flows $(\vec{q}_1, p_1, \Omega_1)$ and $(\vec{q}_2, p_2, \Omega_2)$ to be superposable

$$\text{curl}\,[\vec{q}_1\times\vec{\zeta}_2+\vec{q}_2\times\vec{\zeta}_1] = 0 \tag{15}$$

and for a flow $(\vec{q}_1, p_1, \Omega_1)$ to be self-superposable

$$\text{curl}\,[\vec{q}\times\vec{\zeta}] = 0 \tag{16}$$

The equation of motion for a viscous fluid is

$$\frac{\partial\vec{q}}{\partial t}+\text{grad}\,(\tfrac{1}{2}q^2+\frac{p}{\varrho}+\Omega)+\vec{\zeta}\times\vec{q} = -\nu\,\text{curl}\,\vec{\zeta} \tag{17}$$

Thus the condition for a non-viscous flow to be steady is

$$\text{curl}\,[\vec{\zeta}\times\vec{q}] = 0 \tag{18}$$

but this is same as (16). *Thus the condition for a non-viscous flow to be steady is the same as the condition for a viscous or non-viscous flow to be self-superposable.* The particular cases of this result for two-dimensional and axially symmetric cases were first noticed by Ram Ballabh (1952, 1955) and Prem Prakash (1955).

For two axially symmetric self-superposable flows (16) can be written as

$$\begin{vmatrix} \dfrac{\partial\psi_i}{\partial z} & \dfrac{\partial\psi_i}{\partial r} \\ \dfrac{\partial}{z}\left(\dfrac{\zeta_i}{r}\right) & \dfrac{\partial}{\partial r}\left(\dfrac{\zeta_i}{r}\right) \end{vmatrix} = 0 \qquad [i = 1, 2] \tag{19}$$

Thus ζ_1/r is a function of ψ_1 and ζ_2/r is a function of ψ_2. If two such flows are additive, (7*a*) shows that ψ_1 and ψ_2 are functionally related and then from the above results ζ_1/r and ζ_2/r are also functionally related, so that

$$\psi_2 = f(\psi_1) \tag{20}$$

$$\frac{\zeta_2}{r} = G\left(\frac{\zeta_1}{r}\right) \tag{21}$$

Theorem III. *If two self-superposable axially symmetric flows are superposable, then their stream lines coincide and their vorticities have a relation of the type (21).*

Since steady non-viscous flows are self-superposable, Theorem III also applies to such flows.

Ram Ballabh (1955) had noticed (20) for the case of steady non-viscous flows when one of the motions is irrotational.

A similar theorem can be easily established for two-dimensional self-superposable flows; for if two-dimensional flows $(\vec{q}_1, p_1, \Omega_1)$, $(\vec{q}_2, p_2, \Omega_2)$ are self-superposable

$$(\vec{q}_1\cdot\nabla)\vec{\zeta}_1 = 0,\quad (\vec{q}_2\cdot\nabla)\vec{\zeta}_2 = 0 \tag{22}$$

and if they are superposable on each other

$$(\vec{q}_1\cdot\nabla)\vec{\zeta}_2 = 0 \tag{23}$$

In terms of stream-functions, these can be written as:

$$\frac{\partial(\psi_1, \zeta_1)}{\partial(r, z)} = 0,\quad \frac{\partial(\psi_2, \zeta_2)}{\partial(r, z)} = 0,\quad \frac{\partial(\psi_1, \zeta_2)}{\partial(r, z)} = 0, \tag{24}$$

so that

$$\zeta_1 = f_1(\psi_1),\quad \zeta_2 = f_2(\psi_2),\quad \zeta_2 = f_3(\psi_1) \tag{25}$$

from where we at once deduce

$$\psi_2 = H(\psi_1) \tag{26}$$

$$\zeta_2 = K(\zeta_1) \tag{27}$$

Theorem IV. *If two self-superposable two-dimensional flows are superposable, then stream lines coincide and their vorticities are functionally related.*

The theorem obviously holds for steady non-viscous flows also.

For steady viscous flows, however, (19) is replaced by

$$r\begin{vmatrix} \dfrac{\partial\psi_i}{\partial z} & \dfrac{\partial\psi_i}{\partial r} \\ \dfrac{\partial}{\partial z}\left(\dfrac{\zeta_i}{r}\right) & \dfrac{\partial}{\partial r}\left(\dfrac{\zeta_i}{r}\right) \end{vmatrix} = \nu E^4\psi_i \qquad [i = 1, 2] \tag{28}$$

and for steady viscous flows to be superposable, both (7) and (28) must be satisfied.

4. Decay of vorticity for self-superposable flows. Taking curl of both of (17), we get

$$\frac{\partial}{\partial t}\vec{\zeta} + \text{curl}\,[\vec{\zeta}\times\vec{q}] = \nu\nabla^2\vec{\zeta} \tag{29}$$

For self-superposable flows, using (16), we get the equation for the decay of vorticity as

$$\frac{\partial}{\partial t}\vec{\zeta} = \nu\nabla^2\vec{\zeta} \tag{30}$$

Two-Dimensional flows. For two-dimensional plane flows, it becomes

$$\frac{\partial\zeta}{\partial t} = \nu\nabla^2\zeta \tag{31}$$

where ζ is magnitude of the vorticity vector. This is the well-known Diffusion equation and admits a general integral of the type [Sneddon (1957)]

$$\zeta(x, y, t)\ \sum_{\lambda}\sum_{\mu} C_{\lambda\mu}\cos(\lambda x + \varepsilon\ \ )\cos(\mu y + \varepsilon_\mu)e^{-(\lambda^2+\mu^2)\nu t} \tag{32}$$

If the motion is in circles and ζ is a function of r and t only (31) gives (Lamb 1945)

$$\frac{\partial\zeta}{\partial t} = \nu\left(\frac{\partial^2\zeta}{\partial r^2} + \frac{1}{r}\frac{\partial\zeta}{\partial r}\right) \tag{33}$$

with solutions of the type

$$\left.\begin{aligned} \zeta &= \sum_s \frac{k_s}{4\pi\nu t}e^{-r^2/4\nu t} \\ \text{or}\qquad \zeta &= \sum_s e^{-\nu k_s^2 t}[A_s J_0(k_s r) + B_s Y_0(k_s r)] \end{aligned}\right\} \tag{34}$$

Axi-symmetric flows. For axi-symmetric flows. (30) gives

$$\frac{\partial\zeta}{\partial t} = \nu\left[\nabla^2\zeta - \frac{\zeta}{t^2}\right] \tag{35}$$

which is the same as the equation obtained by Prem Prakash by the cartesian method.

To solve this equation, we use the method of separation of variables *i.e.* assume $\zeta = TRZ$, where T, R, Z are functions of t, r, and z only. This gives

$$\frac{dT}{dt} = -\nu\lambda^2 T,\quad \frac{d^2Z}{dz^2} = \mu^2 Z \tag{36a}$$

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\lambda^2 + \mu^2 - \frac{1}{r^2}\right)R = 0 \tag{36b}$$

Then the general solution comes out to be

$$\zeta = \sum_{\lambda}\sum_{\mu} C_{\lambda\mu}[J_1((\lambda^2+\mu^2)^{\frac{1}{2}}r)]e^{\pm\mu z - \lambda^2\nu t} + \sum_{\lambda}\sum_{\mu} C'_{\lambda\mu}[Y_1((\lambda^2+\mu^2)^{\frac{1}{2}}r)]e^{\pm\mu z - \lambda^2\nu t} \tag{37}$$

In the particular case, when ζ is a function of r and t only, (37) gives

$$\zeta = \sum_{\lambda} e^{-\lambda^2\nu t}[A_\lambda J_1(\lambda r) + B_\lambda Y_1(\lambda r)] \tag{38}$$

Prem Prakash (1955) has also solved (35) in this case and obtained the solution

$$\zeta = e^{-a\nu t}\left\{a_0 r\left(1 - \frac{ar^2}{2\cdot4} + \frac{a^2r^4}{2\cdot4^2\cdot6} - \frac{a^3r^6}{2\cdot4^2\cdot6^2\cdot8} + \dots\right)\right.$$
$$\left. + a_1 r^3\left(1 - \frac{ar^2}{3\cdot5} + \frac{a^2r^4}{3\cdot5^2\cdot7} - \frac{a^3r^6}{3\cdot5^2\cdot7^2\cdot9} + \dots\right)\right\} \tag{39}$$

The first series within the bracket is definitely a solution of (36b), but the second series when substituted in (36b) gives $a_1 = 0$. If the coeffecient of a_0 is denoted by $-2U$, then the second independent solution of (36b) would be

$$U\log a^{\frac{1}{2}}r+\frac{1}{(a)^{\frac{1}{2}}r}\left\{1+\frac{1}{2^2}ar^2-\frac{1}{2^2\cdot 4}\left(\tfrac{2}{2}+\tfrac{1}{4}\right)a^2r^4+\frac{1}{2^2\cdot 4^2\cdot 6}\left(\tfrac{2}{2}+\tfrac{2}{4}+\tfrac{1}{6}\right)a^3r^6+\dots\right\} \tag{40}$$

and thus the solution of (35) independent of z can be written as

$$e^{-\lambda^2\nu t}\left\{(a+b\log r)\left(-\tfrac{1}{2}\lambda r+\frac{1}{2^2\cdot 4}\lambda^3r^3-\frac{1}{2^2\cdot 4^2\, 6}\lambda^5r^5+\dots\right)\right.$$

$$\left.+\frac{b}{\lambda r}\left(1+\frac{1}{2^2}\lambda^2r^2-\frac{1}{2^2\cdot 4}\left(\tfrac{2}{2}+\tfrac{1}{4}\right)\lambda^4r^4+\dots\right)\right\} \tag{41}$$

This is the correct form of (39). This can, however, be more conveniently written as

$$\zeta = e^{-\lambda^2\nu t}[AJ_1(\lambda r)+BY_1(\lambda r)] \tag{42}$$

Flow in a circular pipe. If we consider flow in a circular pipe, then since ζ will be finite at $r = 0$, we take $B = 0$, and in that case

$$u = -Ae^{-\lambda^2\nu t}\int^r J_1(\lambda r)dr+C$$

or

$$u = \frac{A}{\lambda}e^{-\lambda^2\nu t}J_0(\lambda r)+C \tag{43}$$

If $u = u_0e^{-\lambda^2\nu t}$ when $r = 0$, then $C = 0$, $A = u_0\lambda$, and if we assume no slipping at the surface, then $u = 0$ when $r = 0$, so that

$$u_0e^{-\lambda^2\nu t}J_0(a\lambda) = 0 \tag{45}$$

The roots of this equation are (Lamb, 1945)

$$\frac{a\lambda}{\pi} = \cdot 7655,\ 1\cdot 7571,\ 2\cdot 7546,\ \dots \tag{45}$$

and as λ is known for the distribution on the axis, this will determine the possible radii for no slipping at the surface.

The flux across any section at time t is (Sneddon, 1956)

$$u_0e^{-\lambda^2\nu t}\int_0^a 2\pi rJ_0(\lambda r)dr$$

$$= u_0e^{-\lambda^2\nu t}\frac{2\pi a}{\lambda}J_1(\lambda a) \tag{46}$$

We can similarly discuss the non-steady flow between two co-axial circular cylinders.

More general axi-symmetric flow. We now consider the decay of vorticity for the axial symmetric case when the vorticities have both toroidal and poloidal components. This case has been considered by Lakshman Rao (1957) from the point of superposability.

If in the notation of Milne-Thomson (1955)

$$\vec{\zeta} = \zeta_r \vec{i}_r + \zeta_\theta \vec{i}_\theta + \zeta_z \vec{i}_z \tag{47}$$

(30) gives

$$\frac{\partial \zeta_r}{\partial t} = \nu\left(\nabla^2 \zeta_r - \frac{\zeta_r}{r^2} - \frac{2}{r^2}\frac{\partial \zeta_\theta}{\partial \theta}\right) \tag{48a}$$

$$\frac{\partial \zeta_\theta}{\partial t} = \nu\left(\nabla^2 \zeta_\theta - \frac{\zeta_\theta}{r^2} + \frac{2}{r^2}\frac{\partial \zeta_r}{\partial \theta}\right) \tag{48b}$$

$$\frac{\partial \zeta_z}{\partial t} = \nu \nabla^2 \zeta_z \tag{48c}$$

But for axial-symmetric case, ζ_r, ζ_θ, ζ_z are independent of θ, so that these reduce to

$$\frac{\partial \zeta_r}{\partial t} = \nu\left(\nabla^2 \zeta_r - \frac{\zeta_r}{r^2}\right) \tag{49a}$$

$$\frac{\partial \zeta_\theta}{\partial t} = \nu\left(\nabla^2 \zeta_\theta - \frac{\zeta_\theta}{r^2}\right) \tag{49b}$$

$$\frac{\partial \zeta_z}{\partial t} = \nu \nabla^2 \zeta_z \tag{49c}$$

Thus both ζ_r and ζ_θ satisfy (35) for the axi-symmetric case and ζ_z satisfies the equation (31) corresponding to two-dimensional flow. Both these equations have been discussed earlier.

Moreover if

$$\vec{q} = \frac{1}{r}\frac{\partial \psi}{\partial z}\vec{i}_r + \chi \vec{i}_\theta - \frac{1}{r}\frac{\partial \psi}{\partial r}\vec{i}_z. \tag{50}$$

then

$$\zeta_r = -\frac{\partial \chi}{\partial z} \tag{51a}$$

$$\zeta_\theta = \frac{1}{r} E^2 \psi \tag{51b}$$

and

$$\zeta_z = \frac{\chi}{r} + \frac{\partial \chi}{\partial r} \tag{51c}$$

I am grateful to Prof. P. L. Bhatnagar for making me interested in superposability problems and for his general encouragement.

Summary. In the present paper we have (*i*) given some cases where the condition of superposability in axi-symmetry flow is satisfied, (*ii*) established four theorems relating to superposability of two self-superposable flows, (*iii*) studied some non-steady solutions for the decay of vorticity in axi-symmetric case, expressing the solution in terms of Bessel's functions, (*iv*) extended (*iii*) to the case when the velocities can have both toroidal and poloidal components.

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TRANSVERSE COMPONENT OF VELOCITY IN A PLANE SYMMETRICAL FORCED JET OF A COMPRESSIBLE FLUID

By

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1. **Introduction.** M. Ray, in a recent paper (Ray, 1955) obtained a closed analytical solution for a plane forced jet of a compressible fluid under the usual boundary layer conditions. He has obtained analytical expression for u, the velocity component parallel to the jet axis; v the velocity component perpendicular to the jet axis and for the enthalpy i. He has also obtained an equation which determines the stream function ψ. We can obtain an alternative expression for v from a knowledge of this stream function. We find, however, that this does not agree with the value of v which Ray has obtained. We have examined the reason for this and have obtained the correct expression for v. We have used this expression for discussing the variation of v with distance from the jet axis for both forced and unforced jets. In particular we have shown that near the axis, it is always directed away from the axis and at the edges of the jet, it is dircted towards the jet axis.

For incompressible fluids, Ray's results generalise the results of Bickley (1937) for unforced jets, but in this particular case his formula for v does not reduce to that given in (Bickley, 1937). It can be easily verified that our formula does reduce to his. We also see that in all the cases discussed by us the stream lines and the velocity profiles are similar in shape to those sketched by Bickley.

2. **Basic equations.** Using the notation of (Ray, 1955), the basic equations are:

$$\varrho u\frac{\partial u}{\partial x}+\varrho v\frac{\partial u}{\partial y} = c\varrho\frac{u^2}{x}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{1}$$

$$\frac{\partial}{\partial x}(\varrho u)+\frac{\partial}{\partial y}(\varrho v) = 0 \tag{2}$$

$$\varrho u\frac{\partial i}{\partial x}+\varrho v\frac{\partial i}{\partial y} = \frac{\partial}{\partial y}\left(\mu\frac{\partial i}{\partial y}\right)+\mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{3}$$

$$i\varrho = i_1\varrho_1 \tag{4}$$

The assumptions made are:

(*i*) pressure gradient is negligible;

(*ii*) Prandtl number is unity;

(*iii*) $$\frac{\mu}{\mu_1} = \frac{i}{i_1} \tag{5}$$

The boundary conditions are:

on the axis of the jet: $$y = 0,\ v = 0,\ \frac{\partial u}{\partial y} = 0 \tag{6a}$$

at the edge: $$y = \infty, \qquad u = 0 \tag{6b}$$

The transformations used are:

$$\varrho u = \frac{\partial \psi}{\partial y}, \qquad \varrho v = \frac{\partial \psi}{\partial x} \tag{7}$$

$$\eta = (\mu_1 \varrho_1)^{-\frac{1}{2}} x^n \psi, \qquad u = x^m f(\eta) \tag{8}$$

The differential equation for determining $f(\eta)$ is

$$n(f + \eta f') = \frac{d}{d\eta}(ff') \tag{9}$$

where we have taken $$m + 2n = -1, \quad 1 + C + 3n = 0 \tag{10}$$

3. Earlier derivation of v examined Since

$$d\psi = \left(\frac{\partial \psi}{\partial x}\right)_\eta dx + \left(\frac{\partial \psi}{\partial \eta}\right)_x d\eta = \left(\frac{\partial \psi}{\partial y}\right)_x dy + \left(\frac{\partial \psi}{\partial x}\right)_y dy \tag{11}$$

we get on using (7) and (8)

$$-n(\mu_1 \varrho_1)^{\frac{1}{2}} x^{-n-1} \eta\, dx + (\mu_1 \varrho_1)^{\frac{1}{2}} x^{-n} d\eta = \varrho u\, dy - \varrho v\, dx. \tag{12}$$

From (12), the following are deduced in [Ray (1955)]

$$\varrho v = n(\mu_1 \varrho_1)^{\frac{1}{2}} x^{-n-1} \eta \tag{13}$$

$$\varrho u\, dy = (\mu_1 \varrho_1)^{\frac{1}{2}} x^{-n} d\eta \tag{14}$$

Since, however, η is a function of x and y, the correct deductions from (12) would be

$$\varrho v = n(\mu_1 \varrho_1)^{\frac{1}{2}} x^{-n-1} \eta - (\mu_1 \varrho_1)^{\frac{1}{2}} x^{-n} \left(\frac{\partial \eta}{\partial x}\right)_y \tag{15}$$

and $$\varrho u = (\mu_1 \varrho_1)^{\frac{1}{2}} x^{-n} \left(\frac{\partial \eta}{\partial y}\right)_x \tag{16}$$

For (13) and (15) to agree $(\partial \eta / \partial x)_y$ must vanish *i.e.* η should be a function of y alone, which is not correct a priori, and which is not justified by later investigations. In fact if we substitute in (15) for $\partial\eta/\partial x$, the value obtained from (8) and use (7), we get an identity and so such an equation cannot be used to determine v.

In (Ray, 1955), the equations (6*a*) and (13) have been used to obtain the boundary conditions for solving (9) in the form:

when
$$y = 0, \qquad \eta = 0 \tag{17}$$

Fortunately for the plane symmetrical jet in which we are interested, (17) can be obtained from symmetry considerations. By symmetry the axis of the jet is a stream line and ψ is constant on it. Since, however, the numbering of the stream lines is arbitrary, we can choose ψ to be zero on the jet axis, and then from (8), η is zero on the jet axis.

From (16), we get
$$\left(\frac{\partial y}{\partial \eta}\right)_x = \frac{(\mu_1\varrho_1)^{-\frac{1}{2}}x^{-n}}{\varrho u} = \frac{(\mu_1\varrho_1)^{\frac{1}{2}}ix^{-n}}{i_1\varrho_1 u}. \tag{18}$$

Using the expressions obtained for i and u by Ray (1955) viz.
$$i = i_1 - \frac{n^3x^{2m}}{4(2m-n)(m-3n)}(\eta_0^2-\eta^2)\{(2m-n)\eta^2-n\eta_0^2\}. \tag{19}$$

and
$$u = \tfrac{1}{2}(-n)x^m\eta_0^2\left[1-\frac{\eta^2}{\eta_0^2}\right]. \tag{20}$$

and integrating as in [Ray, 1955], we get
$$y = \left(\frac{\mu_1}{\varrho_1}\right)^{\frac{1}{2}} x^{-n-m}\left\{\begin{array}{c} -\dfrac{2}{n\eta_0}\tanh^{-1}\dfrac{\eta}{\eta_0} \\[2ex] +\dfrac{n^2x^{2m}\eta(-3n\eta_0^2+(2m-n)\eta^2)}{6i_1(2m-n)(m-3n)}\end{array}\right\} \tag{21}$$

The R. H. S. of (21) is to include an arbitrary function of x, but this has to vanish, since from (17), when $\eta = 0$, $y = 0$ for all x. Thus we get the same equation as in [Ray, (1955)].

4. New derivation of v. On using (8), we can write (21) as:
$$y = \left(\frac{\mu_1}{\varrho_1}\right)^{\frac{1}{2}} x^{-n-m}\left\{-\frac{2}{n\eta_0}\tanh^{-1}\left(\frac{(\mu_1\varrho_1)^{-\frac{1}{2}}x^n\psi}{\eta_0}\right)+\frac{n^2x^{2m}(\mu_1\varrho_1)^{\frac{1}{2}}x^n\psi}{6i_1(2m-n)(m-3n)}\right.$$
$$\left.\{-3n\eta_0^2+(2m-n)(\mu_1\varrho_1)^{-\frac{1}{2}}x^{2n}\psi^2\}\right. \tag{22}$$

This is the equation to determine the stream lines as well as the stream function ψ as a function of x and y.

From (22) $y(-\psi) = -y(\psi)$, showing again the symmetry of the jet.

If we write (22) as $y = f(x, \psi)$, we get
$$1 = \frac{\partial f}{\partial \psi}\frac{\partial \psi}{\partial y} = \frac{\partial f}{\partial \psi}\varrho u \tag{23}$$

$$0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial \psi}\frac{\partial \psi}{\partial x} = \frac{\partial f}{\partial \psi} + \frac{\partial f}{\partial \psi}(-\varrho v) \tag{24}$$

From (23) and (24) $\qquad v/u = \partial f/\partial x,$

so that
$$\frac{v}{u} = -\frac{2}{n\eta_0}\left(\frac{\mu_1}{\varrho_1}\right)^{\frac{1}{2}} x^{-m-n-1}\left\{-(m+n)\tanh^{-1}\frac{\eta}{\eta_0}+\frac{n\eta\eta_0}{\eta_0^{\,2}-\eta^2}\right\}$$
$$+\left(\frac{\mu_1}{\varrho_1}\right)^{\frac{1}{2}}\frac{n^2x^{m-n-1}\eta}{6i_1(2m-n)(m-3n)}\left\{\begin{matrix}-3mn\eta_0^{\,2}+(2m-n)m\eta^2\\ +(2m-n)2n\eta^2\end{matrix}\right\} \tag{25}$$

Using (10) and (20)
$$v = \left(\frac{\mu_1}{\varrho_1}\right)^{\frac{1}{2}}\tfrac{1}{2}(-n)x^{-1-2n}(\eta_0^{\,2}-\eta^2)\left\{-\frac{2}{n\eta_0}x^n\left((1+n)\tanh^{-1}\frac{\eta}{\eta_0}+\frac{n\eta\eta_0}{\eta_0^{\,2}-\eta^2}\right)\right.$$
$$\left.+\frac{n^2x^{-3n-2}\eta}{6i_1(5n+1)(5n+2)}\begin{bmatrix}3n(1+2n)\eta_0^{\,3}\\ +(5n+2)\eta^3\end{bmatrix}\right\} \tag{26}$$

From (13), we would have got
$$v = \frac{n(\mu_1\varrho_1)^{\frac{1}{2}}x^{-n-1}\eta}{\varrho} = \frac{n(\mu_1\varrho_1)^{\frac{1}{2}}x^{-n-1}}{i_1\varrho_1}i\eta,$$

and using (19), we get
$$v = \frac{n(\mu_1\varrho_1)^{\frac{1}{2}}x^{-n-1}\eta}{i_1\varrho_1}\left\{i_1+\frac{n^3x^{-4n-2}(\eta_0^{\,2}-\eta^2)}{4(5n+1)(5n+2)}\left[(5n+2)\eta^2+n\eta_0^{\,2}\right]\right\} \tag{27}$$

(26) and (27) are obviously different and the difference as explained earlier is due to the non-vanishing of $(\partial\zeta/\partial x)_y$.

5. Derivations from the solution. *Case I. When there is no external force.* In this case [Toose (1955), Kapur (1958)]
$$n = -\tfrac{1}{3},\quad \eta_0 = \sqrt{6}, \tag{28}$$

so that from (26)
$$v = \left(\frac{\mu_1}{\varrho_1}\right)^{\frac{1}{2}}\left\{\frac{1}{\sqrt{6}}x^{-2/3}\left[\tfrac{2}{3}(6-\eta^2)\tanh^{-1}\frac{\eta}{\sqrt{6}}-\tfrac{1}{3}\eta\sqrt{6}\right]\right\}+\frac{1}{216i_1}x^{-4/3}\eta(6-\eta^2)^2$$

or
$$v = \left(\frac{\mu_1}{\varrho_1}\right)^{\frac{1}{2}}\left\{\frac{1}{3\sqrt{6}}\frac{1}{x^{2/3}}\left[6\left(1-\frac{\eta^2}{6}\right)\log\frac{\sqrt{6}+\eta}{\sqrt{6}-\eta}-\eta\sqrt{6}\right]+\tfrac{1}{6}\frac{1}{x^{4/3}}\eta\left(1-\frac{\eta^2}{6}\right)^2\right\} \tag{29}$$

The variation of v in this case with distance from the axis has been fully discussed by Kapur (1958) and it has been shown there that as the distance from the axis increases, the transverse velocity at first is directed away from the axis and is increasing and after attaining a maximum value it decreases and vanishes at a definite distance from the axis. After that till the edge of the jet, the velocity is directed towards the jet axis.

Case II. The external force is constant (= K) along the jet axis. In this case from [Ray, 1955].
$$n = -\tfrac{3}{4},\qquad \eta_0^{\,2} = \tfrac{16}{3}(k/5)^{\frac{1}{2}}, \tag{30}$$

so that
$$v = \left(\frac{\mu_1}{\varrho_1}\right)^{\frac{1}{2}}\left\{\frac{1}{4\eta_0}x^{-\frac{1}{4}}\left(\tfrac{1}{2}(\eta_0^{\,2}-\eta^2)\log\frac{\eta_0+\eta}{\eta_0-\eta}-3\eta\eta_0\right)\frac{9}{9856}x^{3/4}\eta(9\eta_0^{\,2}-14\eta^2)(\eta_0^{\,2}-\eta^2)\right\} \tag{31}$$

As $\eta \to \eta_0$,

$$v \to -\frac{3}{4x^{1/4}}\left(\frac{\mu_1}{\varrho_1}\right)^{1/4}\eta_0. \tag{32}$$

so that, the transverse velocity at the edge of the jet is directed towards the jet axis. Near the jet axis the velocity is directed away from the axis and there will be a curve which will divide the regions in which the velocity is directed away or towards the axis. The equation of the curve is given by

$$\frac{1}{\eta_0 x^{1/4}}\left[\tfrac{1}{2}(\eta_0^2-\eta^2)\log\frac{\eta_0+\eta}{\eta_0-\eta}-3\eta\eta_0\right]+\frac{9}{1694}x^{3/4}\eta(9\eta_0^2-14\eta^2)(\eta_0^2-\eta^2)=0 \tag{33}$$

and

$$y=\left(\frac{\mu_1}{\varrho_1}\right)^{1/4}x^{1/4}\left\{\frac{4}{3\eta_0}\log\frac{\eta_0+\eta}{\eta_0-\eta}+\frac{3}{616}x\eta(9\eta_0^2+7\eta^2)\right\} \tag{34}$$

η being the parameter.

Case III. Flux of momentum is $kx^{5/4}$. In this case from Ray (1955).

$$n=-\tfrac{2}{3},\qquad m=\tfrac{1}{3},\qquad \eta_0^3=\frac{9k}{4(\mu_1\varrho_1)^{1/4}} \tag{35}$$

so that

$$v=\left(\frac{\mu_1}{\varrho_1}\right)^{1/4}\left\{\begin{array}{l}\dfrac{1}{3\eta_0}x^{-1/3}\left((\eta_0^2-\eta^2)\tanh^{-1}\dfrac{\eta}{\eta_0}-2\eta\eta_0\right.\\[2ex] \qquad +\dfrac{\eta x^{1/3}}{189 i_1}(\eta_0^2-2\eta^2)(\eta_0^3-\eta^2)\end{array}\right\} \tag{36}$$

As $\eta \to \eta_0$,

$$v \to -\tfrac{2}{3}\left(\frac{\mu_1}{\varrho_1}\right)^{1/4}\frac{\eta_0}{x^{1/3}} \tag{37}$$

and the variation of velocity with respect to distance from the axis follows a law similar to those followed in the other two cases. The equations of the curve dividing the regions in which velocity is directed away or towards the jet axis are obtained by putting $v=0$ in (36) and $n=-\tfrac{2}{3}$ in (21).

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SOME THEOREMS CONNECTED WITH A GENERALIZED STIELTJES TRANSFORM

By

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1. Introduction.

If

$$f(s) = \int_0^\infty e^{-st}\varphi(t)dt$$

where

$$\varphi(s) = \int_0^\infty e^{-st}\psi(t)dt$$

then

$$f(s) = \int_0^\infty \frac{1}{s+t}\psi(t)dt \tag{1}$$

and $f(s)$ is called the Stieltjes Transform of $\psi(t)$.

Varma (1951) has given a generalization of the Laplace Transform in the form

$$F(s) = \int_0^\infty (st)^{m-\frac{1}{2}} e^{-\frac{1}{2}st} W_{k,m}(st)\varphi(t)dt. \tag{2}$$

where $W_{k,m}$ denotes a Whittaker Function.

He has also shown that if we take $F(s)$ to be the transform of $\varphi(t)$ in the sense of (2) and $\varphi(s)$ to be the ordinary Laplace Transform of $\psi(t)$, then

$$F(s) = \frac{\Gamma(2m+1)}{\Gamma(m-k+\frac{3}{2})}\frac{1}{s}\int_0^\infty {}_2F_1\left[\begin{matrix}2m+1,\ 1\\ m-k+\frac{3}{2}\end{matrix};-\frac{t}{s}\right]\psi(t)dt \tag{3}$$

provided that $\mathbf{Re}\,(2m+1) > 0$.

If $k+m = \frac{1}{2}$, the hypergeometric function in the integral degenerates into a binomial expression and (3) reduces to (1). We will call (3) as the generalized Stieltjes transform of $\psi(t)$.

The object of this paper is to give some theorems connected with the transform (3), which enable us to evaluate certain infinite integrals. The theorems have been illustrated by examples.

2. Theorem 2.1.

If

$$g(u) = \frac{\Gamma(2m+1)}{\Gamma(m-k+\frac{3}{2})}\frac{1}{u}\int_0^\infty {}_2F_1\left[\begin{matrix}2m+1,\ 1\\ m-k+\frac{3}{2}\end{matrix};-\frac{s}{u}\right]\varphi(s)ds$$

and
$$\phi(s) = \int_0^\infty e^{-sv} h(v)\,dv.$$

then
$$g(u) = \frac{1}{u}\int_0^\infty \frac{1}{v} E(2m+1,\, 1,\, 1 : m-k+\tfrac{3}{2} : uv) h(v)\,dv \tag{4}$$

where E denotes the Macrobert function, provided that **Re** $m > 0$, $m-k+\frac{3}{2} \neq 0$ *or a negative integer, and the integrals*

$$\int_0^\infty h(v)\,dv$$

and
$$\int_0^\infty \frac{1}{v} E(2m+1,\, 1,\, 1 : m-k+\tfrac{3}{2} : uv) h(v\, dv$$

converge absolutely.

Proof. We have

$$g(u) = \frac{\Gamma(2m+1)}{\Gamma(m-k+\frac{3}{2})}\frac{1}{u}\int_0^\infty {}_2F_1\left[\begin{matrix}2m+1,\ 1\\ m-k+\frac{3}{2}\end{matrix}; -\frac{s}{u}\right] ds \int_0^\infty e^{-sv} h(v)\,dv$$

$$= \frac{\Gamma(2m+1)}{\Gamma(m-k+\frac{3}{2})}\frac{1}{u}\int_0^\infty h(v)\,dv \int_0^\infty e^{-sv}\, {}_2F_1\left[\begin{matrix}2m+1,\ 1\\ m-k+\frac{3}{2}\end{matrix}; -\frac{s}{u}\right] ds$$

on changing the order of integration,

$$= \frac{\Gamma(2m+1)}{\Gamma(m-k+\frac{3}{2})}\int_0^\infty h(v)\,dv \int_0^\infty e^{-utv}\, {}_2F_1\left[\begin{matrix}2m+1,\ 1\\ m-k+\frac{3}{2}\end{matrix}; -t\right] dt$$

$$= \frac{1}{u}\int_0^\infty \frac{1}{v} E(2m+1,\, 1,\, 1 : m-k+\tfrac{3}{2} : uv) h(v)\,dv$$

provided that **Re** $u \geq 0$, as (Macrobert, 1937-38)

$$\int_0^\infty e^{-x\lambda}\lambda^{\gamma-1}\, {}_2F_1[\alpha, \beta\,; \delta\,; -\lambda]\,d\lambda = \frac{\Gamma(\delta)}{\Gamma(\alpha)\Gamma(\beta)} x^{-\gamma} E(\alpha, \beta, \gamma : \delta : x)$$

provided that **Re** $(x) \geq 0$ and **Re** $(\gamma) > 0$.

To justify the change in the order of integration, let

$$A(s) = {}_2F_1\left[\begin{matrix}2m+1,\ 1\\ m-k+\frac{3}{2}\end{matrix}; -\frac{s}{u}\right] \int_0^\infty e^{-sv} h(v)\,dv$$

and
$$B(v) = h(v)\int_0^\infty e^{-sv}\, {}_2F_1\left[\begin{matrix}2m+1,\ 1\\ m-k+\frac{3}{2}\end{matrix}; -\frac{s}{u}\right] ds.$$

$A(s)$ is uniformly convergent for $s \geq 0$, since

$$\int_0^\infty e^{-sv} h(v)\,dv$$

converges and $B(v)$ is uniformly convergent for $v \geq 0$,

as
$$\int_0^\infty e^{-sv}\, {}_2F_1\left[\begin{matrix}2m+1,\ 1\\ m-k+\frac{3}{2}\end{matrix};-\frac{s}{u}\right]ds$$

converges, since

$$F\left[\begin{matrix}2m+1,\ 1\\ m-k+\frac{3}{2}\end{matrix};-\frac{s}{u}\right] = O(s^{-1})(s\to\infty)(\mathbf{Re}\, m>0)$$
$$= O(1) \quad (s\to 0)$$

Again, if we consider the integral

$$\int_T^\infty h(v)dv \int_{T'}^\infty e^{-sv}F\left[\begin{matrix}2m+1,\ 1\\ m-k+\frac{3}{2}\end{matrix};-\frac{s}{u}\right]ds$$

where T and T' are large, we see that this does not exceed a constant multiple of

$$\int_T^\infty |\,h(v)\,|\,dv \int_{T'}^\infty |\,e^{-sv}s^{-1}\,|\,ds$$

which tends to zero for $\mathbf{Re}\,(v)>0$.

Thus the change in the order of integration is justified under the conditions stated.

Corollary 2, 1. If we put $k+m=\frac{1}{2}$,

$$\frac{1}{uv}E(2m+1,\ 1,\ 1;\ m-k+\tfrac{3}{2};\ uv)$$

reduces to
$$e^{uv}\Gamma(0,\ uv)$$

where $\Gamma(0,\ x)$ denotes the exponential integral

$$\int_x^\infty e^{-t}t^{-1}dt \qquad (x>0)$$

and we have the following known result (Akutowicz, 1948).

If
$$g(u)=\int_0^\infty \frac{1}{u+s}\varphi(s)ds$$

and
$$\varphi(s)=\int_0^\infty e^{-sv}h(v)dv$$

then
$$g(u)=\int_0^\infty e^{uv}\Gamma(0,\ uv)h(v)dv,$$

which is the third iterate of Laplace transform.

The integral (4) may therefore be called a generalization of the third iterate of Laplace transform.

In what follows we will suppose that **Re** $(m) > 0$.

Example 2. 1a. If

$$h(v) = (2v)^{-m+k-3/2}e^{-1/4v}$$

then [BMP, TIT, I, 1954, p. 293 (40)]

$$\varphi(s) = (\sqrt{s})^{m-k+\frac{1}{2}}K_{m-k+\frac{1}{2}}(\sqrt{s})$$

and therefore [BMP, TIT, **II**, 1954, p. 152 (86)]

$$g(u) = 2^{m-k+5/2}u^{m}S_{-2m-1,\,2m}(\sqrt{u})\Gamma(2m+1)$$

provided that

$$\mathbf{Re}\,(u) > 0\,;\quad \mathbf{Re}\,(m-k+\tfrac{3}{2}) > 0.$$

Hence $$\int_0^\infty v^{-m+k-5/2}e^{-1/4v}E(2m+1,\,1,\,1:\,m-k+\tfrac{3}{2};\,uv)dv$$

$$= 2^{2(m-k+2)}\Gamma(2m+1)u^{m+1}S_{-2m-1,\,2m}(\sqrt{u}).$$

provided that $\mathbf{Re}\,u > 0;\qquad \mathbf{Re}\,(m-k+\tfrac{3}{2}) > 0.$

Example 2. 1b. If $$h(v) = \frac{v^{-p}}{\Gamma(1-p)},$$

then $$\varphi(s) = s^{p-1};$$

and [BMP, HTF **I**, 1953, p. 79(4)]

$$g(u) = \frac{\Gamma(p)\Gamma(1-p)\Gamma(2m+1-p)}{\Gamma(m-k+\frac{3}{2}-p)}u^{p-1},$$

provided that $0 < \mathbf{Re}\,p < \min\{\mathbf{Re}\,(2m+1),\,1\};\ m-k+\tfrac{3}{2} \neq 0,\,-1,\,-2,\ldots$

Hence $$\int_0^\infty E(2m+1,1,1;\,m-k+\tfrac{3}{2};\,uv)v^{-p-1}dv$$

$$= \frac{\Gamma(2m+1-p)\Gamma(p)[\Gamma(1-p)]^2}{\Gamma(m-k+\frac{3}{2}-p)}u^{p}.$$

If we put $k+m = \frac{1}{2}$, in the above, we deduce the result

$$\int_0^\infty e^{w}\Gamma(0,w)w^{-p}dw = \Gamma(p)[\Gamma(1-p)]^2.$$

Theorem 2. 2. *If* $$g(u) = \frac{1}{u}\int_0^\infty \frac{1}{v}E\left(\begin{matrix}2m+1,\,1,\,1:\\ m-k+\frac{3}{2}\end{matrix}\,:uv\right)f(v)dv$$

and $$\frac{1}{v}f\left(\frac{1}{v}\right) = \int_0^\infty e^{-vz}\psi(z)dz$$

then
$$g(u) = \frac{1}{u}\int_0^\infty \frac{1}{z}\, E\left(\begin{matrix}2m+1,1,1,1: \\ m-k+\frac{3}{2}\end{matrix} : uz\right)\psi(z)\,dz$$

provided that the integrals involved are uniformly and absolutely convergent.

Proof. We have
$$\begin{aligned} g(u) &= \frac{1}{u}\int_0^\infty \frac{1}{v}\, E\left(\begin{matrix}2m+1,1,1: \\ m-k+\frac{3}{2}\end{matrix}\ uv\right)f(v)\,dv \\ &= \frac{1}{u}\int_0^\infty E\left(\begin{matrix}2m+1,1,1\cdot \\ m-k+\frac{3}{2}\end{matrix} .(u/v)\right)\frac{1}{v}f\left(\frac{1}{v}\right)dv \\ &= \frac{1}{u}\int_0^\infty E\left(\begin{matrix}2m+1,1,1: \\ m-k+\frac{3}{2}\end{matrix} \cdot \frac{u}{v}\right)dv\int_0^\infty e^{-vz}\psi(z)\,dz \\ &= \frac{1}{u}\int_0^\infty \psi(z)\,dz\int_0^\infty E\left(\begin{matrix}2m+1,1,1: \\ m-k+\frac{3}{2}\end{matrix} : \frac{u}{v}\right)e^{-vz}\,dv \end{aligned}$$

on changing the order of integration.

But
$$\int_0^\infty E\left(\begin{matrix}2m+1,1,1\cdot \\ m-k+\frac{3}{2}\end{matrix} : \frac{u}{v}\right)e^{-vz}\,dv = \frac{1}{z}\, E\left(\begin{matrix}2m+1,1,1,1: \\ m-k+\frac{3}{2}\end{matrix} : zu\right)$$

since (Macrobert, 1941)
$$\int_0^\infty e^{-\lambda}\lambda^{\beta-1}E\left(\alpha_1,\ldots,\alpha_p;\,\varrho_1,\ldots,\varrho_q : \frac{x}{\lambda}\right)d\lambda = E(\alpha_1,\ldots,\alpha_p,\beta;\,\varrho_1,\ldots,\varrho_q : x)(\mathbf{Re}\,\beta > 0)$$

Hence
$$g(u) = \frac{1}{u}\int_0^\infty \frac{1}{z}\, E\left(\begin{matrix}2m+1,1,1,1: \\ m-k+\frac{3}{2}\end{matrix} : uz\right)\psi(z)\,dz.$$

provided the integrals involved are uniformly and absolutely convergent.

The following theorems can be easily deduced from Theorem. 2. 2.

Theorem. 2. 2a. *If*
$$\frac{1}{v}f\left(\frac{1}{v}\right) = \int_0^\infty e^{-vz}\psi(z)\,dz,$$

then we have
$$\int_0^\infty \frac{1}{v}\, E\left(\begin{matrix}2m+1\ 1,1 \\ m-k+\frac{3}{2}\end{matrix} : uv\right)f(v)\,dv = \int_0^\infty \frac{1}{z}\, E\left(\begin{matrix}2m+1,1,1,1: \\ m-k+\frac{3}{2}\end{matrix} .uz\right)\psi(z)\,dz.$$

Theorem 2. 2b. *If*
$$g(u) = \frac{1}{u}\int_0^\infty \frac{1}{v_1}E\left(\begin{matrix}2m+1,1,1: \\ m-k+\frac{3}{2}\end{matrix} : uv_1\right)f_1(v_1)\,dv_1,$$
$$\frac{1}{v_1}f_1\left(\frac{1}{v_1}\right) = \int_0^\infty e^{-v_1 v_2}f_2(v_2)\,dv_2,$$
$$\frac{1}{v_2}f_2\left(\frac{1}{v_2}\right) = \int_0^\infty e^{-v_2 v_3}f_3(v_3)\,dv_3,$$

………………………………

and
$$\frac{1}{v_{n-1}}f_{n-1}\left(\frac{1}{v_{n-1}}\right) = \int_0^\infty e^{-v_{n-1}v_n}f_n(v_n)dv_n.$$

then
$$g(u) = \frac{1}{u}\int_0^\infty \frac{1}{v_n}E\left(\begin{matrix}2m+1,1\ldots(n+1)\text{ times}\\ m-k+\frac{3}{2}\end{matrix} ;\, uv_n\right)f_n{}'v_n)dv_n$$

Theorem 2. 2c.
$$\frac{1}{v_1}f_1\left(\frac{1}{v_1}\right) = \int_0^\infty e^{-v_1v_2}f_2(v_2)dv_2$$
$$\frac{1}{v_2}f_2\left(\frac{1}{v_2}\right) = \int_0^\infty e^{-v_2v_3}f_3(v_3)dv_3,$$

……………………………………

and
$$\frac{1}{v_{n-1}}f_{n-1}\left(\frac{1}{v_{n-1}}\right) = \int_0^\infty e^{-v_{n-1}v_n}f_n(v_n)dv_n,$$

then
$$\int_0^\infty \frac{1}{v_1}E\left(\begin{matrix}2m+1,1,1:\\ m-k+\frac{3}{2}\end{matrix} :uv_1\right)f_1(v_1)dv_1 = \int_0^\infty \frac{1}{v_n}E\left(\begin{matrix}2m+1,1\ldots(n+1)\text{ times}:\\ m-k+\frac{3}{2}\end{matrix} :uv_n\right)f_n(v_n)dv_n. \quad (5)$$

We will now give an example of the application of Theorem 2.2c. which is a generalization of Theorem 2. 2a.

Example 2. 2. Let
$$f_n(v_n) = v_n^{\nu}$$
where
$$-1 < \mathbf{Re}\,\nu < 0.$$

Then
$$\frac{1}{v_{n-1}}f_{n-1}\left(\frac{1}{v_{n-1}}\right) = \Gamma(\nu+1)v_{n-1}^{-\nu-1}$$
and
$$f_{n-1}(v_{n-1}) = \Gamma(\nu+1)v^{\nu}_{n-1}.$$

Therefore
$$\frac{1}{v_{n-2}}f_{n-2}\left(\frac{1}{v_{n-2}}\right) = [\Gamma(\nu+1)]^2v_{n-2}^{-\nu-1}$$
and
$$f_{n-2}(v_{n-2}) = [\Gamma(\nu+1)]^2v^{\nu}_{n-2}$$

Continuing this process, we will have
$$f_1(v_1) = [\Gamma(\nu+1)]^{n-1}v_1^{\nu}$$

Hence using (5), we get
$$\int_0^\infty \frac{1}{v_n}E\left(\begin{matrix}2m+1,1\ldots(n+1)\text{ times}:\\ m-k+\frac{3}{2}\end{matrix} :uv_n\right)v_n^{\nu}\,dv_n$$
$$= [\Gamma(\nu+1)]^{n-1}\int_0^\infty \frac{1}{v_1}E\left(\begin{matrix}2m+1,1,1:\\ m-k+\frac{3}{2}\end{matrix} : uv\right)v_1^{\nu}dv_1$$
$$= [\Gamma(\nu+1)]^{n}\frac{\Gamma(2m+1)}{\Gamma(m-k+\frac{3}{2})}\int_0^\infty {}_2F_1\left[\begin{matrix}2m+1,1\\ m-k+\frac{3}{2}\end{matrix} ;\, -\frac{s}{u}\right]s^{-\nu-1}\,ds,$$

using Theorem 2. 1.

$$= [\Gamma(\nu+1)]^{n+1}\frac{\Gamma(-\nu)\Gamma(2m+1+\nu)u^{-\nu}}{\Gamma(m-k+\frac{3}{2}+\nu)}$$

Theorem 2. 3. *If* $g(u)$ *is the generalized Stieltjes transform of* $\varphi(s)$ *i.e. if*

$$g(u) = \frac{\Gamma(2m+1)}{\Gamma(m-k+\frac{3}{2})}\frac{1}{u}\int_0^\infty {}_2F_1\left[\begin{matrix}2m+1,\ 1\\ m-k+\frac{3}{2}\end{matrix};-\frac{s}{u}\right]\varphi(s)ds,$$

and $s^{-m+k-\frac{1}{2}}\varphi(s)$ *is the generalized Stieltjes transform of* $h(x)$ *in the sense*

$$s^{-m+k-\frac{1}{2}}\varphi(s) = \int_0^\infty \frac{1}{(s+x)^\sigma}h(x)dx,$$

then
$$g(u) = \frac{\Gamma(2m+1)\Gamma(m+k-\frac{1}{2}+\sigma)\Gamma(-m+k-\frac{1}{2}+\sigma)}{\Gamma(m+k+\frac{1}{2}+\sigma)\Gamma(\sigma)}u^{\frac{1}{2}-\sigma+m-k}$$
$$\times\int_0^\infty {}_2F_1\left[\begin{matrix}m+k-\frac{1}{2}+\sigma,\ -m+k-\frac{1}{2}+\sigma\\ m+k+\frac{1}{2}+\sigma\end{matrix};1-\frac{x}{u}\right]h(x)dx,$$

provided that **Re** $(m-k+\frac{3}{2})>0$, **Re** $(m+k-\frac{1}{2}+\sigma)>0$,

Re $(-m+k-\frac{1}{2}+\sigma)>0$ *and* $h(x)$ *is such that* $\int_0^\infty \frac{1}{(s+x)^\sigma}h(x)dx$ *converges absolutely.*

Proof. We have

$$g(u) = \frac{\Gamma(2m+1)}{\Gamma(m-k+\frac{3}{2})}\frac{1}{u}\int_0^\infty F\left[\begin{matrix}2m+1,\ 1\\ m-k+\frac{3}{2}\end{matrix};-\frac{s}{u}\right]s^{m-k+\frac{1}{2}}ds\int_0^\infty\frac{1}{(s+x)^\sigma}h(x)dx$$

$$= \frac{\Gamma(2m+1)}{\Gamma(m-k+\frac{3}{2})}\frac{1}{u}\int_0^\infty h(x)dx\int_0^\infty F\left[\begin{matrix}2m+1,\ 1\\ m-k+\frac{3}{2}\end{matrix};-\frac{s}{u}\right]\frac{s^{m-k-\frac{1}{2}}}{(s+x)^\sigma}ds$$

on changing the order of integration.

But [BMP, TIT, II 1954, p. 400 (10)].

$$\int_0^\infty F\left[\begin{matrix}2m+1,\ 1\\ m-k+\frac{3}{2}\end{matrix};-\frac{s}{u}\right]\frac{1}{(s+x)^\sigma}s^{m-k+\frac{1}{2}}ds$$

$$= u^{m-k-\sigma+3/2}\int_0^\infty F\left[\begin{matrix}2m+1,\ 1\\ m-k+\frac{3}{2}\end{matrix};-v\right]\left(v+\frac{x}{u}\right)^{-\sigma}v^{m-k+\frac{1}{2}}dv$$

$$= u^{m-k-\sigma+3/2}\frac{\Gamma(m+k-\frac{1}{2}+\sigma)\Gamma(-m+k-\frac{1}{2}+\sigma)\Gamma(m-k+3/2)}{\Gamma(m+k+\frac{1}{2}+\sigma)\Gamma(\sigma)}$$
$$\times\,{}_2F_1\left[\begin{matrix}m+k-\frac{1}{2}+\sigma,\ -m+k-\frac{1}{2}+\sigma\\ m+k+\sigma+\frac{1}{2}\end{matrix};1-\frac{x}{u}\right],$$

provided that **Re** $(m-k+3/2)>0$, **Re** $(m+k+\sigma-\frac{1}{2})>0$,

and
$$\mathbf{Re}\,(-m+k-\tfrac{1}{2}+\sigma) > 0.$$

Therefore
$$g(u) = \frac{\Gamma(2m+1)\Gamma(m+k-\frac{1}{2}+\sigma)\Gamma(-m+k-\frac{1}{2}+\sigma)}{\Gamma(m+k+\frac{1}{2}+\sigma)\Gamma(\sigma)} u^{\frac{1}{2}-\sigma+m-k}$$
$$\times \int_0^\infty {}_2F_1\left[\begin{matrix} m+k-\frac{1}{2}+\sigma, & -m+k-\frac{1}{2}+\sigma \\ & m+k+\frac{1}{2}+\sigma \end{matrix}\; ; 1-\frac{x}{u}\right] dx.$$

The change in the order of integration can be justified in a manner similar to that of Theorem 2.1 under the conditions stated in the theorem.

Corollary 2. 3a. If $m-k+\frac{1}{2} = 0$ and $2m+1 = \varrho$, then we have
$$g(u) = \Gamma(\varrho) u^{\rho-1} \int_0^\infty \frac{1}{(u+s)^\rho} \varphi(s) ds,$$

and
$$\varphi(s) = \int_0^\infty \frac{1}{(s+x)^\sigma} h(x) dx.$$

Hence
$$g(u) = \frac{\Gamma(\varrho)}{(\varrho+\sigma-1)} u^{-\sigma} \int_0^\infty {}_2F_1\left[\begin{matrix} \varrho+\sigma-1, \ \sigma \\ \varrho+\sigma \end{matrix}\; ; 1-\frac{x}{u}\right] h(x) dx$$

provided that $\mathbf{Re}\,\sigma > 0$ and $\mathbf{Re}\,(\varrho+\sigma-1) > 0$.

Corollary 2. 3b. If we further put $\varrho = \sigma = 1$ in Corollary 2.3a

we have
$$g(u) = \int_0^\infty \frac{1}{u+s} \varphi(s) ds$$

and
$$\varphi(s) = \int_0^\infty \frac{1}{s+x} h(x) dx$$

In this case
$$g(u) = \frac{1}{u} \int_0^\infty F\left[\begin{matrix} 1, \ 1 \\ 2 \end{matrix}\; ; 1-\frac{x}{u}\right] h(x) dx$$
$$= \int_0^\infty \frac{1}{x-u} \log\left(\frac{x}{u}\right) h(x) dx$$

which is the second iterate of the Stieltjes transform (Widder, 1941).

Example 2.3a. Let $\sigma = 1$ and
$$h(x) = (-\tfrac{1}{2}) x^{3/2(m-k+\frac{1}{2})} \{\cos[(m-k+3/2)\pi] J_{m-k+\frac{1}{2}}(\sqrt{x})$$
$$+ \sin[(m-k+3/2)\pi] Y_{m-k+\frac{1}{2}}(\sqrt{x})\}$$

then [BMP, TIT. II, 1954, p. 229 (33)
$$\varphi(s) = (\sqrt{s})^{m-k+\frac{1}{2}} K_{m-k+\frac{1}{2}}(\sqrt{s}),$$

and [BMP, TIT, II, 1954, p. 152 (86)]
$$g(u) = \Gamma(2m+1) 2^{m-k+5/2} u^m S_{-2m-1, 2m}(\sqrt{u}),$$

where $S_{\mu,\nu}(z)$ represents the Lommel's function defined by

$$S_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\,{}_1F_2\left[1\,;\frac{\mu+\nu+3}{2},\ \frac{\mu-\nu+3}{2}\ ;-\frac{z^2}{4}\right]$$

$$+2^{\mu-1}\Gamma\left(\frac{\mu-\nu+1}{2}\right)\Gamma\left(\frac{\mu+\nu+1}{2}\right)\left[\sin\left(\frac{\mu-\nu}{2}\pi\right)J_\nu(z)-\cos\left(\frac{\mu-\nu}{2}\pi\right)Y_\nu(z)\right].$$

provided that **Re** $(m-k+\frac{3}{2})>0$ and **Re** $(m-k)<0$.

Hence
$$\int_0^\infty F\left[\begin{matrix} m+k+\frac12,\ -m+k+\frac12 \\ m+k+\frac32\end{matrix}\ ;1-\frac{x}{u}\right](\sqrt{x})^{3(m-k+\frac12)}$$
$$\times\{\cos[(m-k+\tfrac32)\pi]J_{m-k+\frac12}(\sqrt{x})+\sin(m-k+\tfrac32)\pi]Y_{m-k+\frac12}(\sqrt{x})\}dx$$
$$=-2^{m-k+5/2}\frac{(m+k+\frac12)}{\Gamma(-m+k+\frac12)}u^{k+\frac12}S_{-2m-1,2m}(\sqrt{u}).$$

Example 2.3b. Let
$$h(x)=\frac{\Gamma(\sigma)}{\Gamma(p+\sigma+1)\Gamma(1-p)}x^{p+\sigma-2}$$

then
$$\varphi(s)=s^{p+m-k-\frac12}$$

and [BMP, HTF, **I**, 1953, p. 79(4)

$$g(u)=\frac{\Gamma(m+k-p+\frac12)\Gamma(m-k+p+\frac12)\Gamma(-m+k-p+\frac12)}{\Gamma(1-p)}u^{p+m-k-\frac12},$$

provided that **Re** $(p+\sigma-1)>0$, **Re** $(1-p)>0$

$$0<\textbf{Re}\,(p+m-k+\tfrac12)<1\,;\ m-k+\tfrac32\neq 0,\ -1,\ -2,\ldots$$

Hence
$$\int_0^\infty F\left[\begin{matrix} m+k-\frac12+\sigma,\ -m+k-\frac12+\sigma \\ m+k+\frac12+\sigma\end{matrix}\ ;1-\frac{x}{u}\right]x^{p+\sigma-2}dx$$
$$=\frac{\Gamma(m+k-p+\frac12)\Gamma(m-k+p+\frac12)\Gamma(-m+k-p+\frac12)\Gamma(m+k+\sigma+\frac12)\Gamma(p+\sigma-1)}{\Gamma(2m+1)\Gamma(m+k-\frac12+\sigma)\Gamma(-m+k-\frac12+\sigma)}u^{p-1+\sigma}$$

under the conditions stated above.

I am indebted to Dr. K. M. Saksena for guidance and help in the preparation of this paper.

D. S. B. GOVERNMENT COLLEGE
NAINITAL, U.P.

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CALCUTTA MATHEMATICAL SOCIETY.

Report of the Council for the year 1958 to the Annual General Meeting of the Society.

The Council of the Calcutta Mathematical Society has the pleasure to submit the following report on the general concerns of the Society for the year 1958 as required by the provisions of Rule 25 of the Society's constitution.

The Council. The Council of the Society for the year 1958 consisting of the officers and other members elected at the last Annual General Meeting together with the Assistant Secretary and Editorial Secretary, was constituted as follows:

President
S. N. Bose

Vice-Presidents
V. V. Narlikar,
Ram Behari,
P. C. Mahalanobis,
B. N. Prasad,
N. R. Sen,

Treasurer
S. C. Ghosh,

Secretary
T. C. Roy,

Assistant Secretary
J. De,

Editorial Secretary
P. K. Ghosh,

Other Members of the Council
A. C. Banerji,
H. M. Sengupta,
G. Bandyopadhyay,
C. N. Srinvasiengar
N. M. Basu,
P. P. Chattarji,
D. D. Mallick,
S. K. Basu
P. L. Bhatnagar,
U. R. Burman,
J. De.
N. A. Shastri

General. On the 21st January, 1958, Prof. S. Nikolsky of U.S.S.R. delivered an interesting and informative lecture on "Theory of Approximation of functions by Polynomials and Variational method" in the room of the Calcutta Mathematical Society at 8 P.M.

On the 29th January, 1958 in the Annual General Meeting the retiring President, Prof. V. V. Narlikar, in his learned presidential address, gave a very clear picture about the modern trends in researches on the Theory of Relativity. He also pointed out clearly and briefly the possible lines of the developement of the Theory of Relativity. His address was published in the Bulletin of the Society (Vol. 50, pp. 1-8).

Along with the various usual activities of the Society, the Golden Jubilee of the Society has been celebrated at the completion of fifty years in a beifitting manner. The various functions were spread over the period from the 25th December, 1958 to the 31st of January, 1959.

Delegations. In reply to an invitation from the Indian Mathematical Society the Council nominated the following members to represent the Society at the Session of the Golden Jubilee and Twenty-fourth Conference of the Indian Mathematical Society:—

Prof. Ram Behari—Professor of Mathematics, Delhi University.
Dr. B. N Prasad—Reader in Mathematics, Allahabad University.
Dr. G. Bandyopadhpay—Assistant Professor of Applied Mathematics, I. I. T., Kharagpur.
Sri J. De—Assistant Secretary, Calcutta Mathematical Society.

In the special occasion of the Golden Jubilee of the Indian Mathematical Society our President on behalf of the Society, sent a message of good wishes to the Indian Mathematical Society.

The Council nominated the following members to represent the Society at the forthcoming Session (1959) of the Indian Science Congress at New Delhi:

Prof. M. Ray—Professor of Mathematics and Dean Faculty of Science, Agra University.
Prof Ram Behari—Professor of Mathematics. Delhi University.
Prof. S. K. Chakraborty—Professor of Mathematics, B. E. College, Howrah.
Prof. N. L. Ghosh—Professor of Mathematics. Presidency College, Calcutta.
Dr. B. N. Prasad—Reader in Mathematics, Allahabad University.
Prof. R. N. Sen—Head of the Department of Pure Mathematics, Calcutta University.
Prof. B. B. Sen—Head of the Department of Mathematics, Jadavpur University.

The Council nominated the following members to represent the Society at the International Congress of Mathematicians, Edinburgh 1958.

Prof. S. N. Bose—President of the Society and Vice-Chancellor, Viswa Bharati University.
Dr. G. Bandyopadhyay—Assistant Professor of Mathematics, I. I. T., Kharagpur.
Dr. S. C. Das—Dominion Observatory, Ottawa.

Membership. The Council desires to report that during the year under review eighteen new names have been added to the list of Seciety's members including one life member.

Meetings during 1958. The Council held nine meetings during the year and there were five General Meetings devoted to the reading of original papers communicated to the Society for publications in the Bulletin. In connection with the Golden Jubilee Celebrations, there was a special general meeting held on the 28th December, 1958. Prof. Humayun Kabir, the Minister for Scientific Research and Cultural Affairs, Government of India, kindly attended the meeting as the Chief Guest. Prof. Kabir expressed his great concern at the low standard of the teaching of Mathematics in India and suggested that overhaul moves should be taken by the Societies or similar bodies and the leading mathematicians to raise the standard of the teaching of Mathematics. In the absence of the President, Prof. S. N. Bose, one of the Vice-Presidents Prof. N. R. Sen presided over the meeting. Prof. N. K. Sidhanta, Vice-Chancellor, Calcutta University was also present.

Ammendment of Rule. On the recomendation of the Council the following alteration of Rule 14 of the existing Rules and Regulations of the Society was made by majority votes of the Members of the Society.

That the following lines of Rule 14 "the annual contribution shall be twelve Rupees in the case of Members residing in Calcutta and its Suburbs and Howrah (who shall be called Resident Members), and six rupees in the case of Non-resident Members" be replaced by the lines "the annual contribution shall be twelve rupees in the case of ordinary members".

Publications. During the year under review the Society has published five issues of the Bulletin, *viz.*, Vol. 49, Nos. 3-4 and Vol. 50, Nos. 1-3. The Council takes this opportunity to record here the Society's indebtedness to the authorities of the Calcutta University for printing the Bulletin free of charge and to the Officers and members of the Staff of the University Press for their valued services.

The Council is glad to announce that a good number of papers from eminent Mathematicians of the world have been received in connection with the Golden Jubilee of the Society and those will be published in a special publication in a Commemoration Volume of the Society. In this connection the Council is glad to announce that the Calcutta University will arrange to publish the special Golden Jubilee Volume free of cost.

The Council considered the letter from the Assistant Registrar, Calcutta University to the Secretary, Calcutta Mathematical Society regarding the publication of the Bulletin of the Society in its several meetings with deep concern.

A reply to that letter has already been sent to the Calcutta University appreciating with gratitude the help and encouragement which the Calcutta University with its motto of 'Advacement of Learning' has rendered to the Calcutta Mathematical Society by nourishing it and by publishing its Bulletin for the long stretch of nearly half a century free of charge. The Society has always considered the University of Calcutta as its most benevolent patron and will continue to think so in future with an earnest hope that the Society will not be deprived of the patronage and the University of Calcutta will continue the printing of its Bulletin without cost as in the past.

Exchange of Publications. The transmission of the Society's publications to various countries of the world has been carried on regularly during the year and two new exchange relations have also been established. For various reasons, we have lost one exchange relation.

Library. The Council desires to report that during the year under review it was with extreme financial hardship that the Society was somehow able to maintain uninterrupted the run of the research periodicals which were being regularly subscribed for the past few years and that this could be made possible only by an entire suspension of the purchase of recently published reference books and important newly published research periodicals. This indeed amounts to a great denial of research facilities to the members of the Society but it had to be imposed under the exigencies of circumstances.

We should earnestly hope that our National Government (Central and Provincial) would appreciate the necessity of mathematical researches in the country at this present juncture of scientific development of the world and render their substancial help to the Society for this purpose.

The Council expresses hearty thanks to Sarbasree D. C. Kar, J. C. Kar and R. C. Kar for their benevolent gift of the valued collections of their late father, Dr S. C. Kar which has been ceremonially received by the Society on the 28th December, 1958 in the presence of Prof. Humayun Kabir, the Minister of Scientific Research and Cultural Affairs, Government of India. The collection has been preserved as 'Dr. S C. Kar Collection', as a part of the Library of the Society.

Finance. The annual accounts of the Society have been presented to the Council in the standardized form by the auditors Messrs. S. K. Dutt and B. K. Lahiri. The Council offers them its sincere thanks for their honorary services. The Council also takes this opportunity to make a few observations on the audited statement of acounts during the year under review. A glance at the receipt and expenditure sides of the statement at once reveals an extremely gloomy picture regarding the financial position of the Society. Though we received some more grants during the year under review from the Government of West Bengal than that of last year, we have not received any grant from the Government of India. The balance at Banks [the State Bank of India (General Fund) and the United Bank of India] which we can spend for our general expenses have almost been reduced to the minimum balances required to be deposited at the Banks. As regards the other balances *e.g.* K. K. G. P. and C. E. Cullis Funds we cannot spend them as those are endowment funds meant for specific purposes. Though the price of the Bulletin and subscription of non resident members have been increased and more income has been there out of these, still to cope with the day-to-day increasing market price almost all the amount has been exhausted. The Council, with great regret, is obliged to repeat the last years verdict that the Society is almost on the verge of a collapse. Unless the Society receives liberal grants-in-aid from the Governments the Council fears that the Society, which perhaps is the oldest Mathematical Research organisation in India will have to face a very hard time in pursuing its aims and objects.

The Society received the following grants during the year under review:

(i)	Government of West Bengal	Rs 2,000/-
(ii)	National Institute of Sciences of India	Rs. 1,000/-

and the Council offers its grateful thanks to the Government of West Bengal and the National Institute of Sciences of India for these grants.

CALCUTTA MATHEMATICAL SOCIETY

RECEIPTS AND DISBURSEMENTS ACCOUNTS OF THE CALCUTTA MATHEMATICAL SOCIETY FOR THE YEAR ENDING 31ST. DECEMBER, 1958.

Receipts.

		Rs. nP.	Rs. nP.
1. Opening Balance			
(a) With Secretary			
(i) In cash	2.89		
(ii) In stamps	7.26	10.15	
(b) Balance at Banks			
(i) State Bank of India (Gen. Fund)	348.15		
(ii) Do (K K G P Fund) ..	933.10		
(iii) Do (C E Cullis Fund) ...	478 00		
(iv) United Bank of India	529 57		
Do (in suspense)	12 00		
(v) Postal Savings Bank	537 12	2,837.94	
(c) G. P Notes (General Fund) (Face value Rs 6,000)		5,663.72	
(d) G. P. Notes (K. K. G. P Fund) (Face value Rs. 2,000)		1,937.48	
(e) G. P. Notes (C. E. Cullis Fund) (Face value Rs. 8,000)		6,558.06	17,007.35
2. Membership subscription	1,513.75		
3. Admission fees . ..	180.00		
4. Sale proceeds	2,335 66		
5. Donation	100.00	4,129.41	
6. Grants			
(i) Govt. of West Bengal ...	2,000.00		
(ii) Nat. Inst. of Sciences.	1,000.00	3,000.00	
7 Interest			
(a) G. P Notes (General Fund) ...	180.00		
(b) Do (K. K. G P. Fund)	60.00		
(c) Do (C. E Cullis Fund)	240.00		
(d) Postal Savings Accounts	18.38	498.38	7,627.79
Total ... Rs.			24,635.14

Disbursements

		Rs. nP.	Rs. nP.
1 Establishment		354 03	
2 Meetings		183 68	
3. Books & Journals (including binding)		1,906.12	
4. Bulletins			
(a) Establishment	944.16		
(b) Compositors' Salary	2,060.00		
(c) Papers, Blocks, Types etc	1,243.44		
(d) Postage	626.90		
(e) Conveyance (Press)	42.09	4,916.59	
5. Printing & Stationery		130.14	
6. Postage (General)		135 07	
7. Bank charges		42.66	
8. Miscellaneous (including conveyance charges)		78 67	7,746.96
9. Closing Balance			
(a) With Secretary			
(i) In cash	4.27		
(ii) In stamps	0.54	4.81	
(b) Balance at Banks			
(i) State Bank of India (Gen. Fund)	459 30		
(ii) Do (K. K. G. P Fund) .	992.60		
(iii) Do C. E. Cullis Fund) ...	717.00		
(iv) United Bank of India	475.13		
Do (in suspense)	44.58		
(v) Postal Savings	35 50	2,724.11	
(c) G. P Notes (General Fund) (Face value Rs 6,000)		5,663 72	
(d) Do (K. K. G P. Fund) (Face value Re 2,000)		1,937 48	
(e) (C. E Cullis Fund) Face value Rs 8,000)		6,558.06	16,888.18
Total ... Rs.			24,635.14

To
The Members of the Calcutta Mathematical Society,

We have examined the above accounts with the Books and Vouchers relating thereto and certify it to be correctly drawn up therefrom and in accordance with the information and explanations given to us.

S. K DUTT
B. K. LAHIRI } *Auditors.*

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A CLASS OF STEADILY INCREASING CONTINUOUS FUNCTIONS WHICH ARE NOT ABSOLUTELY CONTINUOUS

By

H. M. Sengupta and P. L. Ganguli, *Calcutta.*

(Received November 25, 1957—Revised March 7, 1958.)

The classical example of a function $f(x)$ continuous and monotone in $0 \leqq x \leqq 1$, which is not an integral, is well known and is to be found in text books. The function, usually called Cantor function, was studied by Hille E and Tamarkin T. D. (1929). In defining the function, numbers in $[0, 1]$ are exhibited in the scales of 3 and 2. Gilman (1932) generalised the above function by exhibiting numbers in the scales of α and β where α and β are positive integers such that $\alpha-1$ is divisible by $\beta-1$. This process, therefore, gives rise to a class of Cantor functions which has the power of the set of positive integers and which contains the classical example of Cantor's function. A Cantor function was defined by P. Herzog and B. H. Bissinger (1948) by using the representation of real numbers as simple continued fractions. J. von Neumann (1950) gives an example of a steadily increasing continuous function which is not absolutely continuous. It contains a key to the construction of a class of such functions. The method, however, is primarily geometric in nature. The present article is devoted to writing down a class of infinite series $\Sigma a_n = 1$ $(a_n > 0)$ and then defining a function of the above type corresponding to each such series in an arithmetical way. This class has the power of the continuum and the classical Cantor function appears as a degenerate case.

We start with an arbitrary convergent series Σb_n of positive terms such that $\Sigma b_n = C < 1$. We then construct another series of positive terms Σa_n such that

$$(1) \quad \Sigma a_n = 1 \qquad (2) \quad a_n > R_n = a_{n+1} + a_{+2} + \dots$$

Some properties of such series were studied by Prof. P. Kesava Menon (1948).

We impose on a_n the further restriction

$$(3) \quad 0 < a_n - R_n < b_n/2^{n-1}.$$

With the help of each series of the above type we propose to construct a Cantor function.

An actual construction of a series of the above type may be carried out as follows.

We choose a_1 arbitrarily such that

$$\tfrac{1}{2} < a_1 < (1+b_1)/2$$

Once a_1 has been chosen, we choose a_2 such that

$$\frac{1-a_1}{2} < a_3 < \min\left\{ \frac{1-a_1}{2} + \frac{b_1}{2.2},\ 1-a_1 \right\}$$

When a_2 has been choosen, we chose a_3 such that

$$\frac{1-a_1-a_2}{2} < a_3 < \min\left\{ \frac{1-a_1-a_2}{2} + \frac{b_3}{2^3},\ 1-a_1-a_2 \right\}$$

When $a_1, a_2, \ldots, a_{n-1}$ have been chosen in this manner, we choose a_n such that

$$\frac{1-a_1-a_2-\ldots-a_{n-1}}{2} < a_n < \min\left\{ \frac{1-a_1-a_2-\ldots-a_{n-1}}{2} + \frac{b_n}{2^n},\ 1-a_1-a_2-\ldots-a_{n-1} \right\}$$

This process being continued indefinitely, we get a series Σa_n of positive terms such that

(1) $\Sigma a_n = 1$

(2) $a_n > R_n$

(3) $0 < a_n - R_n < b_n/2^{n-1}$.

Proof: Since $a_n < 1-a_1-a_2-\ldots-a_{n-1}$

we have $a_1+a_2+\ldots+a_n < 1$

Hence Σa_n is convergent and $\Sigma a_n \leqq 1$

Also $a_n \to 0$ as $n \to \infty$

Again $2a_n > 1-a_1-a_2-\ldots-a_{n-1}$

∴ $\therefore\ a_1+a_2+\ldots+a_n > 1-a_n$

Hence $\Sigma a_n > 1$ showing that $\Sigma a_n = 1$

Now

$$R_n = a_{n+1}+a_{n+2}+\ldots$$
$$= 1-(a_1+a_2+\ldots+a_n)$$

Hence from (1) $a_n > 1-(a_1+a_2+\ldots+a_n) = R_n$

Also $1-a_1-a_2-\ldots-a_{n-1} = a_n + R_n$

Next $a_n < \dfrac{1-a_1-a_2-\ldots-a_{n-1}}{2} + \dfrac{b_n}{2^n}$

∴ $2a_n < a_n + R_n + b_n/2^{n-1}$

Hence $a_n - R_n < b_n/2^{n-1}$

It may be noted that the numbers a_n may be taken all rational if we so like and then R_n is also rational. They are at our disposal and subject to the conditions laid down we can choose them in an infinite variety of ways. As a matter of fact, the aggregate of the series of the above type, for each choice of the series Σb_n, has the power of the continuum.

Having constructed the series Σa_n we define a function $f(x)$ on $0 \leqq x \leqq 1$, according to the following plan.

Let S be the Cantor set on $0 \leqq x \leqq 1$. The elements of S in the ternary scale are expressible in the form

$$\cdot d_1 d_1 \ldots d_n \ldots$$

where each d is either 0 or 2.

The set CS consists of points of the unit interval whose representation in ternary scale consists of expressions of the form

$$\cdot \varrho_1 \varrho_2 \ldots \varrho_n \ldots$$

where at least one ϱ is 1.

It may be noted that numbers where

$$\varrho_k = 1, \quad \varrho_{k+1} = \varrho_{k+2} = \ldots = 0$$

and $\varrho_1, \varrho_2, \ldots \varrho_{k-1}$ are 0 or 2 only, belong to S.

We, however, agree to express numbers of S by using the digits 0 and 2 only. This will ensure unicity of representation of points of S.

To any point of S whose representation is of the form

$$x = \cdot d_1 d_2 \ldots d_n \ldots$$

where each d_n is 0 or 2, we associate the number

$$f(x) = \Sigma a'_n$$

where $a'_n = 0$ if $d_n = 0$ and $a'_n = a_n$ if $d_n = 2$.

We define $f(x)$ on CS as follows:

Let $\delta_1, \delta_2, \delta_3, \ldots$ be the everywhere dense set of enumerable open intervals of CS. Any interval δ_k is such that its end points ξ and η, $(\xi < \eta)$ belong to S and such that

$$\xi = \cdot u'_1 u'_2 \ldots u'_{m-1}\, 0\, 222$$
$$\eta = \cdot u'_1 u'_2 \ldots u'_{m-1}\, 2\, 000$$

where $u'_1, u'_2, \ldots u'_{m-1}$ are 0 or 2.

Hence $\qquad f(\xi) = a'_1 + a'_2 + \ldots + a'_{m-1} + a_{m+1} + a_{m+2} + \ldots$

$\qquad\qquad f(\eta) = a'_1 + a'_2 + \ldots + a'_{m-1} + a_m$

Obviously, $f(\xi) < f(\eta)$ since $a_m > R_m$

We define $f(x)$ in the interior of δ_k by

$$f(x) = f(\xi) + (x - \xi)\frac{f(\eta) - f(\xi)}{\eta - \xi}$$

It is easy to show that $f(x)$ is steadily increasing and continuous in $0 \leqq x \leqq 1$.

Now, from the construction of $f(x)$ on the complementary intervals constituting CS, it follows that on

$$\tfrac{1}{3} < x < \tfrac{2}{3}, \quad f'(x) = 3(a_1 - R_1)$$

On each of the following complementary intervals, which we shall refer to as intervals of the second order *i.e.* intervals deleted at the 2nd stage to produce S

viz. (i) $\frac{1}{9} < x < \frac{2}{9}$ (ii) $\frac{7}{9} < x < \frac{8}{9}$

we have $f'(x) = 3^2(a_2 - R_2)$

On each of the intervals of the 3rd order *viz.*

(i) $\frac{1}{3^3} < x < \frac{2}{3^3}$, (ii) $\frac{7}{3^3} < x < \frac{8}{3^3}$, (iii) $\frac{19}{3^3} < x < \frac{20}{3^3}$, (iv) $\frac{25}{3^3} < x < \frac{26}{3^3}$

we have $f'(x) = 3^3(a_3 - R_3)$

On each of the 2^3 complementary intervals of the 4th order, we have

$$f'(x) = 3^4(a_4 - R_4).$$

In general, on each of 2^{n-1} complementary intervals of the n^{th} order, we have

$$f'(x) = 3^n(a_n - R_n)$$

The function $f(x)$ so defined is thus continuous and steadily increasing in $0 \leqq x \leqq 1$ and $f'(x)$ is summable in $0 \leqq x \leqq 1$. Further,

$$\begin{aligned} \int_0^1 f'(x)dx &= (a_1 - R_1) + 2(a_2 - R_2) + 2^2(a_3 - R_3) + \dots \\ &< b_1 + b_2 + b_3 + \dots \\ &= C < 1 = f(1) - f(0) \end{aligned}$$

It, therefore, follows that each series Σa_n constructed in the above way gives rise to a function continuous and steadily increasing on $0 \leqq x \leqq 1$, which is not absolutely continuous.

Since the aggregate of such series has the power of the continuum, the aggregate of such functions also has the power of the continuum.

In particular, if we take

$$\begin{aligned} a_1 &= \tfrac{1}{2} \\ a_2 &= (1 - a_1)/2 = \tfrac{1}{4} \\ a_3 &= (1 - a_1 - a_2)/2 = \tfrac{1}{8} \text{ and so on} \end{aligned}$$

then the above process gives us, as a degenerate case, the classical example of a continuous monotone non-decreasing function $f(x)$ on $0 \leqq x \leqq 1$, such that $f(x)$ is not an integral.

DEPARTMENT OF PURE MATHEMATICS,
CALCUTTA UNIVERSITY.

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SELF-RECIPROCAL FUNCTION AND $\varpi_{\mu,\nu}$-TRANSFORM

BY

KRISHNA JI SRIVASTAVA, *Lucknow.*

(Communicated by Secretary—Received April 8, 1957)

1. The object of this paper is to apply the Erdélyi-Kober (Erdélyi, 1940; Kober, 1940) operators of the fractional integration to develope the theory of $\varpi_{\mu,\nu}$-transform. The operators of the fractional integration and differention for the class $L_2(0, \infty)$ are defined as

$$I^+_{\eta,\alpha} f = \frac{z^{-\eta-\alpha}}{\Gamma(\alpha)} \int_0^z (z-t)^{\alpha-1} t^{\eta} f(t)\, dt \tag{1.1}$$

$$K^-_{\eta,\alpha} f = \frac{z^{\eta}}{\Gamma(\alpha)} \int_z^{\infty} (t-z)^{\alpha-1} t^{-\eta-\alpha} f(t)\, dt \tag{1.2}$$

$$I^-_{\eta,\alpha} f = \frac{z^{-\eta-\alpha}}{\Gamma(\alpha)} \int_z^{\infty} (t-z)^{\alpha-1} t^{\eta} f(t)\, dt \tag{1.3}$$

$$K^+_{\eta,\alpha} f = \frac{z^{\eta}}{\Gamma(\alpha)} \int_0^z (z-t)^{\alpha-1} t^{-\eta-\alpha} f(t)\, dt \tag{1.4}$$

where **Re** $\eta > -\frac{1}{2}$, **Re** $\alpha > 0$. The kernels bave to be 'cut' when **Re** $\eta < -\frac{1}{2}$. When **Re** $\alpha = 0$ the operators still exist as mean square limits and in particular

$$I^+_{\eta,0} f = K^-_{\eta,0} f = f. \tag{1.5}$$

For negative values of **Re** α we define $I^+_{\eta,\alpha}$ as $(I_{\eta+\alpha,-\alpha})^{-1}$ and $K^-_{\eta,\alpha}$ as $(K^-_{\eta,+\alpha,-\alpha})^{-1}$.

Bhatnagar (1954) has proved in details that the kernel

$$\varpi_{\mu,\nu}(x) = x^{\frac{1}{2}} \int_0^{\infty} J_\nu(t) J_\mu(x/t)\, dt/t, \tag{1.6}$$

plays the role of a transform, and calls a function $g(x)$ to be the $\varpi_{\mu,\nu}$–transform of $f(x)$ if it satisfies the integral equation

$$g(x) = \int_0^{\infty} \varpi_{\mu,\nu}(xy)\, f(y)\, dy \quad (\textbf{Re}\ \mu, \nu > -\tfrac{1}{2}) \tag{1.7}$$

In case $g(x) = f(x)$ we call $f(x)$ to be $R_{\mu,\nu}$.

The Mellin Transform $\mathfrak{M}_t f$ of a function $f(x) \in L_p(0, \infty)$ $(1 \leq p \leq 2)$ is defined by

$$\mathfrak{F}(s) \equiv \mathfrak{M}_t f(x) = \int_0^{\infty} x^{it} f(x) dx \quad (p = 1) \tag{1.8}$$

and

$$\mathfrak{F}(s) \equiv \mathfrak{M}_t f(x) = \underset{x\to\infty}{\overset{\text{index } p'}{\text{l. i. m.}}} \int_{1/x}^{x} f(x) x^{it-1/p'} dx \quad (1<p\leq 2) \tag{1.9}$$

where $1/p+1/p'=1$.

The inverse Mellin transform $\mathfrak{M}^{-1}\Phi(t)$ of a function $\in L'_p.(-\infty, \infty)(1\leq p'\leq 2)$ is defined by

$$\mathfrak{M}^{-1}\varphi(t) = \tfrac{1}{2}\pi \int_{-\infty}^{\infty} \varphi(t) x^{-it} dt \qquad (p'=1) \tag{1.10}$$

$$\mathfrak{M}^{-1}\varphi(t) = \tfrac{1}{2}\pi \underset{T\to\infty}{\overset{\text{index } p}{\text{l. i. m.}}} \int_{-T}^{T} \varphi(t) x^{-it-1/p} dt, \quad (1<p'\leq 2) \tag{1,11}$$

2. Hardy and Titchmarsh (1937) have defined a class of functions $A(\omega, a)$, where $0<\omega\leq\pi$, $a>\frac{1}{2}$ as functions which (i) are analytic functions of $x=re^{i\theta}$, regular in the angle A defined by $r>0$, $|\theta|<\omega$ and (ii) are $O[|x|^{-a-\delta}]$ for the small x and $O[|x|^{a-1+\delta}]$ for large x, for every positive δ and uniformly in any angle $|\theta|\leq\omega-\varepsilon<\omega$. Bhatnagar (1958) has given the following theorem:

A necessary and sufficient condition that a function $f(x)$ of $A(\omega, a)$ should be $R_{\mu,\nu}$ is that it should be of the form

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^s \Gamma(\tfrac{1}{2}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}s) x^{-s} \psi(s)\, ds \tag{2.1}$$

where $\psi(s)$ is regular in $a<\sigma<1-a$; $(s\equiv\sigma+it)$ satisfies the functional equation $\psi(s)=\psi(1-s)$ and is

$$O[e^{(\frac{1}{2}\pi-\omega+\epsilon)|t|}]$$

for every positive ϵ and uniformly in any strip interior to $a<\sigma<1-a$ and c is any value of σ in $a<\sigma<1-a$.

3. **Theorem 1.** *If $\{I^+_{\eta,\alpha}\}^m\{K^-_{\eta,\alpha}\}^n$ stands for the operator $I^+_{\eta,\alpha}$ performed m times and $K^-_{\eta,\alpha}$ performed n times in all, in any order, and if $I^+_{\eta,\alpha}$ and $K^-_{\eta,\alpha}$ belong to $L_2(0, \infty)$ then $\{I^+_{\eta,\alpha}\}^m\{K^-_{\eta,\alpha}\}^n+\{I^+_{\eta,\alpha}\}^n\{R^-_{\eta,\alpha}\}^m$ is an operator transforming an $R_{\mu,\nu}$ into another $R_{\mu,\nu}$ function.*

Proof: Let, for the sake of simplicity, $f(x)$ belonging to the class $L_2(0, \infty)$ be $R_{\mu,\nu}$. Then* we have

---

*A necessary and sufficient condition that *a function $f(x) \in L_2(0, \infty)$ be $R_{\mu,\nu}$ is that it should be of the form*

$$f(x) = \frac{1}{2\pi} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} 2^{\frac{1}{2}+it}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\mu+it)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}it)\psi(\tfrac{1}{2}+it) x^{-\frac{1}{2}-it} dt$$

the integral is a mean square integral, where

$$\mathfrak{F}(\tfrac{1}{2}+it) = 2^{\frac{1}{2}+it}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\mu+\tfrac{1}{2}it)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}it)\psi(\tfrac{1}{2}+it)$$

belongs to $L_2(-\infty, \infty)$ and $\psi(\frac{1}{2}+it)\equiv\psi(\frac{1}{2}-it)$ i.e., $\psi(\frac{1}{2}+it)$ is an even function.

$$f(x) = \int_{\frac12 - i\infty}^{\frac12 + i\infty} P(\tfrac12 + it) x^{-\frac12 - it}\, dt,$$

where
$$P(\tfrac12 + it) = \frac{1}{2\pi} 2^{\frac12 + it} \Gamma(\tfrac12 + \tfrac12\mu + \tfrac12 it)\Gamma(\tfrac12 + \tfrac12\nu + \tfrac12 it)\psi(\tfrac12 + it)$$

and $\psi(\frac12 + it)$ satisfies the conditions

$$\psi(\tfrac12 + it) = \psi(\tfrac12 - it).$$

and
$$\psi(\tfrac12 - it) = O(e^{(\frac12\pi - \omega + \epsilon)\,|t|}).$$

Now
$$I^{+}_{\eta,\alpha} f(x) = \frac{x^{-\eta-\alpha}}{\Gamma(\alpha)} \int_0^x (x-p)^{\alpha-1} p^{\eta}\, dp \int_{\frac12 - i\infty}^{\frac12 + i\infty} P(\tfrac12 + it) p^{-\frac12 - it} dt$$

$$= \frac{x^{-\eta-\alpha}}{\Gamma(\alpha)} \int_{\frac12 + i\infty}^{\frac12 + i\infty} P(\tfrac12 + it)\, dt \int_0^x (x-p)^{\alpha-1} p^{\eta - \frac12 - it}\, dp$$

$$= \int_{\frac12 - i\infty}^{\frac12 + i\infty} P(\tfrac12 + it) x^{-\frac12 - it} \frac{\Gamma(\eta + \frac12 - it)}{\Gamma(\alpha + \eta + \frac12 - it)} dt. \tag{3.1}$$

The change in the order of integration in (3.1) as valid for **Re** $\eta > -\frac12$; **Re** $\alpha > 0$, on account of the absolute convergence of both p-and t-integrals, the latter by virtue of the asymptotic behaviour (Whittaker and Watson, 1952, p. 279)

$$\Gamma(\alpha + i\beta + i\gamma t) = O(e^{-\frac12\pi t\,|\gamma|}\, t^{\alpha-1}) \text{ as } |t| \to \infty$$

$$|P(\tfrac12 + it) t^{-\frac12 - it}| = O(e^{(-\omega + \epsilon)\,|t|}) \text{ for all } \epsilon > 0.$$

Repeating the process it is easy to see that, in general

$$\{I^{+}_{\eta,\alpha}\}^m f(x) = \int_{\frac12 - i\infty}^{\frac12 + i\infty} P(\tfrac12 + it) x^{-\frac12 - it} \left[\frac{\Gamma(\eta + \frac12 - it)}{\Gamma(\alpha + \eta + \frac12 - it)} \right]^m dt. \tag{3.2}$$

Similarly

$$K^{-}_{\eta,\alpha} f(x) = \frac{x^{\eta}}{\Gamma(\alpha)} \int_x^{\infty} (p-x)^{\alpha-1} p^{-\eta-\alpha} dp \int_{\frac12 - i\infty}^{\frac12 + i\infty} P(\tfrac12 + it) p^{-\frac12 - it} dt$$

$$= \int_{\frac12 - i\infty}^{\frac12 + i\infty} P(\tfrac12 + it) x^{-\frac12 - it} \frac{\Gamma(\eta + \frac12 + it)}{\Gamma(\alpha + \eta + \frac12 + it)}\, dt \tag{3.3}$$

provided **Re** $\alpha > 0$, **Re** $\eta > -\frac12$. The change in the order of integration being justified as before. Repeating the above process n times it easily follows that

$$\{I^{+}_{\eta,\alpha}\}^m \{K^{-}_{\eta,\alpha}\}^n f(x) = \int_{\frac12 - i\infty}^{\frac12 + i\infty} P(\tfrac12 + it) x^{-\frac12 - it} \left[\frac{[\Gamma(\eta + \frac12 - it)]^m [\Gamma(\eta + \frac12 + it)]^n}{[\Gamma(\alpha + \eta + \frac12 - it)]^m [\Gamma(\alpha + \eta + \frac12 + it)]^n} \right] dt \tag{3.4}$$

provided that **Re** $\alpha > 0$, **Re** $\eta > -\frac12$.

Hence

$$[\{I^{+}_{\eta,\alpha}\}^{m}\{K^{-}_{\eta,\alpha}\}^{n}+\{I^{+}_{\eta,\alpha}\}^{n}\{K^{-}_{\eta,\alpha}\}^{m}]f(x)=\frac{1}{2\pi}\int_{\frac12-i\infty}^{\frac12+i\infty}2^{\frac12+it}\,\Gamma(\tfrac12+\tfrac12\nu+\tfrac12 it)\Gamma(\tfrac12+\tfrac12\mu+\tfrac12 it)\times$$

$$\psi_1(\tfrac12+it)x^{-\frac12-it}dt \qquad (3.5)$$

where

$$\psi_1(\tfrac12+it)=\left[\frac{[\Gamma(\eta+\frac12-it)]^{m}[\Gamma(\eta+\frac12+it)]^{n}}{[\Gamma(\alpha+\eta+\frac12-it)]^{m}\,[\Gamma(\alpha+\eta+\frac12+it)]^{n}}\right.$$

$$\left.+\frac{[\Gamma(\eta+\frac12-it)]^{n}[\Gamma(\eta+\frac12+it)]^{m}}{[\Gamma(\alpha+\eta+\frac12-it)]^{n}[\Gamma(\alpha+\eta+\frac12+it)]^{m}}\right]\psi(\tfrac12+it)$$

Since $\psi_1(\frac12+it)$ satisfies the condition $\psi_1(\frac12+it)=\psi_1(\frac12-it)$ the theorem follows.

Theorem 2. *If $\{I^{-}_{\eta,\alpha}\}^{m}\{K^{+}_{\eta,\alpha}\}^{n}$ stands for the operators $I^{-}_{\eta,\alpha}$ performed m times and $K^{+}_{\eta,\alpha}$ performed n times in all, in any order, then*

$$\{I^{-}_{\eta,\alpha}\}^{m}\{K^{+}_{\eta,\alpha}\}^{n}+\{I^{-}_{\eta,\alpha}\}^{n}\{K^{+}_{\eta,\alpha}\}^{m}$$

is an operator transforming an $R_{\mu,\nu}$ into another $R_{\mu,\nu}$ function.

Corollaries. (*i*) When $m=n$ we have from theorem 1:

If $f(x)$ is $R_{\mu,\nu}$ then $\{I^{+}_{\eta,\alpha}\}^{m}\{K^{-}_{\eta,\alpha}\}^{m}f(x)$ is also $R_{\mu,\nu}$, provided **Re** $\eta>-\frac12$, **Re** $\alpha>0$.

(*ii*) When $\alpha=1$ we find that the fractional operators reduce to

$$I^{+}_{\eta,1}=x^{-\eta-1}\int_{0}^{x}y^{\eta}\,dy \quad \text{and}$$

$$K^{-}_{\eta,1}=x^{\eta}\int_{x}^{\infty}y^{-\eta-1}dy,$$

and we obtain a result similar to that given by Bhatnagar (1953). Taking $\mu=\nu/2-\frac12$, $\nu=\nu/2+\frac12$ we get a result similar to one given by Gupta (1945, Theorem 1).

Examples: We give here two examples for the case $m=n=1$.

(*i*) Taking $\psi(s)=2^{\frac12\mu-\frac12\nu-\frac12}/\Gamma(\frac12+\frac12\nu+\frac12 s)\Gamma(\frac12+\frac12\nu-\frac12 s)$

we obtain from (2.1)

$$f(x)=x^{\frac12(\mu-\nu+1)}J_{\frac12(\mu+\nu)}(x), \text{ an } R_{\mu,\nu} \text{ function.}$$

Then,

$$I^{+}_{\eta,\alpha}f(x)=(\tfrac12)^{\frac12(\mu+\nu)}\,x^{\mu+\frac12}\;\frac{\Gamma(\eta+\mu+\frac32)}{\Gamma(\frac12\mu+\frac12\nu+1)\;\Gamma(\alpha+\eta+\mu+\frac32)}\times$$

$${}_2F_3\left[\begin{matrix}\frac12\eta+\frac12\mu+\frac34,\ \frac12\eta+\frac12\mu+\frac54\,; \\ \frac12\mu+\frac12\nu+1,\ \frac12\alpha+\frac12\eta+\frac12\mu+\frac34,\ \frac12\alpha+\frac12\eta+\frac12\mu+\frac54\,;\end{matrix}\ -x^2/4\right]$$

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References to literature in the text should be given, whenever possible, in chronological order, only the names of the authors and years of publication, in brackets, being given. They should be cited in full at the end of paper, the author's name following form, *viz.*, name or names of authors; year of publication; name of the journal (abbreviation); number of volume; and lastly, the page number. The following would be a useful illustration:

Wilson, B. M. (1922), *Proc. Lond. Math. Soc.* (2), **21**, 285.

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Hence, by Theorem 1)

$$\{K^{-}_{\eta,\alpha}\}\{I^{+}_{\eta,\alpha}\}f(x) = 2^{\frac{1}{2}\mu-\frac{1}{2}\nu-\eta-\frac{1}{2}}\frac{\Gamma(1+2\eta)\Gamma(-\frac{1}{2}\eta+\frac{1}{2}\mu+\frac{1}{4})x^{\eta}}{\Gamma(\frac{1}{2}\mu+\frac{1}{2}\eta+\frac{1}{4})\Gamma(\alpha+1+2\eta)\Gamma(\alpha)}\times$$

$$ {}_4F_5\left[\begin{matrix}\eta+\frac{1}{2},\ \eta+1,\ 1-\frac{1}{2}\alpha,\ \frac{1}{2}-\frac{1}{2}\alpha\,; \\ \frac{1}{2}\nu+\frac{1}{2}\eta+\frac{3}{4},\ \frac{1}{2}+\frac{1}{2}\alpha+\eta,\ 1+\frac{1}{2}\alpha+\eta,\ \frac{1}{2},\ \frac{3}{4}+\frac{1}{2}\eta-\frac{1}{2}\mu\,;\end{matrix}\ -x^2/4\right]$$

$$+2^{\frac{1}{2}\mu-\frac{1}{2}\nu-\eta-5/2}\frac{\Gamma(2+2\eta)\Gamma(-\frac{1}{2}\eta+\frac{1}{2}\mu+\frac{1}{4})x^{\eta+1}}{\Gamma(\frac{1}{2}\nu+\frac{1}{2}\eta+\frac{5}{4})\Gamma(\alpha+2+2\eta)\Gamma(\alpha-1)}\times$$

$$ {}_4F_5\left[\begin{matrix}\eta+1,\ \eta+\frac{3}{2},\ \frac{3}{2}-\frac{1}{2}\alpha,\ 1-\frac{1}{2}\alpha\,; \\ \frac{1}{2}\nu+\frac{1}{2}\eta+\frac{5}{4},\ \frac{1}{2}\alpha+\eta+1,\ \frac{1}{2}\alpha+\eta+\frac{3}{2},\ \frac{3}{2},\ \frac{5}{4}+\frac{1}{2}\eta-\frac{1}{2}\mu\,;\end{matrix}\ -x^2/4\right]$$

$$+2^{-\frac{1}{2}\mu-\frac{1}{2}\nu}\frac{\Gamma(\eta+\mu+\frac{3}{2})\Gamma(\eta-\mu-\frac{1}{2})x^{\mu+\frac{1}{2}}}{\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu+1)\Gamma(\alpha+\eta+\mu+\frac{3}{2})\Gamma(\alpha+\eta+\mu-\frac{1}{2})}\times$$

$$ {}_4F_5\left[\begin{matrix}\frac{1}{2}\eta+\frac{1}{2}\mu+\frac{3}{4},\ \frac{1}{2}\eta+\frac{1}{2}\mu+\frac{5}{4},\ \frac{3}{4}-\frac{1}{2}\alpha-\frac{1}{2}\eta+\frac{1}{2}\mu,\ \frac{5}{4}-\frac{1}{2}\alpha-\frac{1}{2}\eta+\frac{1}{2}\mu\,; \\ \frac{1}{2}\mu+\frac{1}{2}\nu+1,\ \frac{1}{2}\alpha+\frac{1}{2}\eta+\frac{1}{2}\mu+\frac{3}{4},\ \frac{1}{2}\alpha+\frac{1}{2}\eta+\frac{1}{2}\mu+\frac{5}{4},\ \frac{3}{4}-\frac{1}{2}\eta+\frac{1}{2}\mu,\ \frac{5}{4}-\frac{1}{2}\eta+\frac{1}{2}\mu\,;\end{matrix}\ -x^2/4\right]$$

is also an $R_{\mu,\nu}$ function.

(*ii*) Next, taking $\psi(s) = 2^{\frac{1}{2}\mu+\frac{1}{2}\nu-3/2}$

we obtain from (2.1), $f(x) = x^{\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}}K_{\frac{1}{2}\nu-\frac{1}{2}\mu}(x)$, an $R_{\mu,\nu}$ function.

Hence

$$\{K^{-}_{\eta,\alpha}\}\{I^{+}_{\eta,\alpha}\}f(x) = 2^{\frac{1}{2}\mu-\frac{1}{2}\nu-1}\frac{\Gamma(\frac{1}{2}\mu-\frac{1}{2}\nu)\Gamma(\eta+\nu+\frac{3}{2})\Gamma(\eta-\nu-\frac{1}{2})^{\nu+\frac{1}{2}}}{\Gamma(\alpha+\nu+\eta+\frac{3}{2})\Gamma(\alpha+\eta-\nu-\frac{1}{2})}$$

$$ {}_4F_5\left[\begin{matrix}\frac{1}{2}\eta+\frac{1}{2}\nu+\frac{3}{4},\ \frac{1}{2}\eta+\frac{1}{2}\nu+\frac{5}{4},\ -\frac{1}{2}\eta+\frac{1}{2}\nu-\frac{1}{2}\alpha+\frac{5}{4},\ -\frac{1}{2}\eta+\frac{1}{2}\nu-\frac{1}{2}\alpha+\frac{3}{4}\,; \\ \frac{1}{2}\alpha+\frac{1}{2}\eta+\frac{1}{2}\nu+\frac{3}{4},\ \frac{1}{2}\alpha+\frac{1}{2}\eta+\frac{1}{2}\nu+\frac{5}{4},\ 1-\frac{1}{2}\mu+\frac{1}{2}\nu,\ \frac{5}{4}-\frac{1}{2}\eta+\frac{1}{2}\nu,\ \frac{3}{4}-\frac{1}{2}\eta+\frac{1}{2}\nu\,;\end{matrix}\ -x^2/4\right]$$

$$+2^{\frac{1}{2}\nu-\frac{1}{2}\mu-1}\frac{\Gamma(\frac{1}{2}\nu-\frac{1}{2}\mu)\Gamma(\eta+\mu+\frac{3}{2})\Gamma(\eta-\mu-\frac{1}{2})}{\Gamma(\alpha+\mu+\eta+\frac{3}{2})\Gamma(\alpha+\eta-\mu-\frac{1}{2})}x^{\mu+\frac{1}{2}}\times$$

$$ {}_4F_5\left[\begin{matrix}\frac{1}{2}\eta+\frac{1}{2}\mu+\frac{3}{4},\ \frac{1}{2}\eta+\frac{1}{2}\mu+\frac{5}{4},\ -\frac{1}{2}\eta+\frac{1}{2}\mu-\frac{1}{2}\alpha+\frac{5}{4},\ -\frac{1}{2}\eta+\frac{1}{2}\mu-\frac{1}{2}\alpha+\frac{3}{4}\,; \\ \frac{1}{2}\alpha+\frac{1}{2}\eta+\frac{1}{2}\mu+\frac{3}{4},\ \frac{1}{2}\alpha+\frac{1}{2}\eta+\frac{1}{2}\mu+\frac{5}{4},\ 1-\frac{1}{2}\nu+\frac{1}{2}\mu,\ \frac{5}{4}-\frac{1}{2}\eta+\frac{1}{2}\mu,\ \frac{3}{4}-\frac{1}{2}\eta+\frac{1}{2}\mu\,;\end{matrix}\ -x^2/4\right]$$

$$+2^{\frac{1}{2}\mu+\frac{1}{2}\nu-3/2-\eta}\frac{\Gamma(\frac{1}{2}\nu-\frac{1}{2}\eta+\frac{1}{4})\Gamma(\frac{1}{2}\mu-\frac{1}{2}\eta+\frac{1}{4})\Gamma(1+2\eta)}{\Gamma(1+\alpha+2\eta)\Gamma(\alpha)}x^{\eta}\times$$

$$ {}_4F_5\left[\begin{matrix}\eta+\frac{1}{2},\ \eta+1,\ 1-\frac{1}{2}\alpha,\ \frac{1}{2}-\frac{1}{2}\alpha\,; \\ \frac{1}{2}\alpha+\eta+\frac{1}{2},\ \frac{1}{2}\alpha+\eta+1,\ \frac{1}{2},\ \frac{3}{4}-\frac{1}{2}\nu+\frac{1}{2}\eta,\ \frac{3}{4}-\frac{1}{2}\mu+\frac{1}{2}\eta\,;\end{matrix}\ -x^2/4\right]$$

$$+2^{\frac{1}{2}\mu+\frac{1}{2}\nu-5/2-\eta}\frac{\Gamma(\frac{1}{2}\nu-\frac{1}{2}\eta-\frac{1}{4})\Gamma(\frac{1}{2}\mu-\frac{1}{2}\eta-\frac{1}{4})\Gamma(2+2\eta)}{\Gamma(2+\alpha+2\eta)\Gamma(\alpha-1)}x^{\eta+1}\times$$

$$ {}_4F_5\left[\begin{matrix}\eta+1,\ \eta+\frac{3}{2},\ \frac{3}{2}-\frac{1}{2}\alpha,\ 1-\frac{1}{2}\alpha\,; \\ \frac{1}{2}\alpha+\eta+1,\ \frac{1}{2}\alpha+\eta+\frac{3}{2},\ \frac{3}{2},\ \frac{5}{4}+\frac{1}{2}\nu-\frac{1}{2}\eta,\ \frac{5}{4}+\frac{1}{2}\mu-\frac{1}{2}\eta\,;\end{matrix}\ -x^2/4\right]$$

is an $R_{\mu,\nu}$ function.

4. In this section I consider the transform in a modified form which has been defined in one of my previous papers (Srivastava, 1957); i.e.,

$$\mathfrak{S}_{\mu,\nu}\varphi \equiv F(x) = \lim_{N\to\infty}. \text{Sq.} \int_0^N (xy)^{-\frac{1}{4}} \varpi_{\mu,\nu}(4\sqrt{xy})\, \varphi(y)\, dy \tag{4.1}$$

where $\varphi(y) \varepsilon L_2(0.\ \infty)$.

Putting $\varphi(y) = (2y)^{-\frac{1}{4}} f(2y^{\frac{1}{2}})$; $F(y) = (2x)^{-\frac{1}{4}} g(2x^{\frac{1}{2}})$, (4.1) reduces to (1.7). Further, we see that $f \varepsilon L_2(0, \infty)$ implies $F \varepsilon L_2(0, \infty)$ because $|\varphi|_2 = |f|_2$ and $|F|_2 = |g|_2$*.

Now let us consider a pair of functions consisting of an arbitrary function and its transform of order $(2\mu, 2\nu)$ and then consider operations changing this pair of functions into another pair of functions connected by the transform of some other order $(2\xi, 2\eta)$. The class of operations obtained thus include certain results given by Bhatnagar (1953).

Now I formulate two rules for the functions belonging to $L_2(0, \infty)$.

Rule 1. Let $F = \mathfrak{S}_{2\mu,2\nu} f$, $\mathfrak{K} = \mathfrak{M}K$, $\mathfrak{T} = \mathfrak{M}k$, then

$$\Gamma(\mu+\tfrac{1}{2}-it)\Gamma(\nu+\tfrac{1}{2}-it)\Gamma(\xi+\tfrac{1}{2}-it)\Gamma(\eta+\tfrac{1}{2}-it)\mathfrak{K}(t) =$$
$$\Gamma(\mu+\tfrac{1}{2}+it)\Gamma(\nu+\tfrac{1}{2}+it)\Gamma(\xi+\tfrac{1}{2}+it)\Gamma(\eta+\tfrac{1}{2}+it)\mathfrak{T}(-t) \tag{4.2}$$

is a necessary and sufficient conditions that

$$\int_0^\infty K(xy)F(y)\, dy = \mathfrak{S}_{2\xi,2\eta}\left\{\int_0^\infty k(xy)f(y)\, dy\right\} \tag{4.3}$$

Rule 2. Let $F = \mathfrak{S}_{2\mu,2\nu} f$, $\mathfrak{K} = \mathfrak{M}K$, $\mathfrak{T} = \mathfrak{M}k$, then

$$\Gamma(\mu+\tfrac{1}{2}+it)\,\Gamma(\nu+\tfrac{1}{2}+it)\,\Gamma(\xi+\tfrac{1}{2}-it)\,\Gamma(\eta+\tfrac{1}{2}-it)\,\mathfrak{K}(t) =$$
$$\Gamma(\mu+\tfrac{1}{2}-it)\,\Gamma(\nu+\tfrac{1}{2}-it)\,\Gamma(\xi+\tfrac{1}{2}+it)\,\Gamma(\eta+\tfrac{1}{2}+it)\,\mathfrak{T}(-t) \tag{4.4}$$

is a necessary and sufficient condition that

$$\int_0^\infty K(x/y)\, F(y)\, dy = \mathfrak{S}_{2\xi,2\eta}\left\{\int_0^\infty k(x/y)\, f(y)\, dy\right\} \tag{4.5}$$

Proof of the Rule. Necessity: Taking the Mellin transform of both sides of (4.3), we get

$$\int_0^\infty x^{-\frac{1}{2}+it}\, dt \int_0^\infty K(xy)F(y)\, dy = \int_0^\infty x^{-\frac{1}{2}+it}\, dx\left[\mathfrak{S}_{2\xi,2\eta}\int_0^\infty k(xy)f(y)\, dy\right]$$

i.e., $$\int_0^\infty F(y)\, dy \int_0^\infty K(xy)x^{-\frac{1}{2}+it} dx = \int_0^\infty x^{-\frac{1}{2}+it} dx\left[\int_0^\infty \varpi_{2\xi,2\eta}(4\sqrt{xy})(xy)^{-\frac{1}{4}} dy \int_0^\infty k(zy)f(z)dz\right]$$

---

* $|f|_2$ means $\left[\int_0^\infty |f(t)|^2\, dt\right]^{\frac{1}{2}}$

i.e., $\mathfrak{R}(t)\int_0^\infty F(y)y^{-\frac{1}{2}-it}dy = \int_0^\infty x^{-\frac{1}{2}+it}dx\int_0^\infty f(z)\,dz\int_0^\infty \varpi_{2\xi,\,2\eta}(4\sqrt{xy})(xy)^{-\frac{1}{2}}k(zy)dy$

i.e., $\dfrac{\Gamma(\mu+\frac{1}{2}-it)\Gamma(\nu+\frac{1}{2}-it)}{\Gamma(\mu+\frac{1}{2}+it)\Gamma(\nu+\frac{1}{2}+it)}\mathfrak{R}(t)\mathfrak{M}_t f = \int_0^\infty f(z)dz\int_0^\infty k(zy)\,dy\int_0^\infty \varpi_{2\xi,\,2\eta}(4\sqrt{xy})(xy)^{-\frac{1}{2}}x^{-\frac{1}{2}+it}dx$

$$= \int_0^\infty f(z)\,dz\int_0^\infty k(zy)\,dy\;y^{-\frac{1}{2}-it}\frac{\Gamma(\xi+\frac{1}{2}+it)\,\Gamma(\eta+\frac{1}{2}+it)}{\Gamma(\xi+\frac{1}{2}-it)\,\Gamma(\eta+\frac{1}{2}-it)}$$

i.e., $$\frac{\Gamma(\mu+\frac{1}{2}-it)\Gamma(\nu+\frac{1}{2}-it)}{\Gamma(\mu+\frac{1}{2}+it)\Gamma(\nu+\frac{1}{2}+it)}\mathfrak{R}(t)\mathfrak{M}_t f = \frac{\Gamma(\xi+\frac{1}{2}+it)\Gamma(\eta+\frac{1}{2}+it)}{\Gamma(\xi+\frac{1}{2}-it)\Gamma(\eta+\frac{1}{2}-it)}t(-t)\mathfrak{M}_t f.$$

where the change in the order of integrations are permissible by the absolute convergence since $f(x)$, $K(x)$ and $k(x)$ belong to $L_2(0,\infty)$ and $\varpi_{\mu,\nu}(x) = O(x^{-\frac{1}{2}})$ for $x\to\infty$ and $\varpi_{\mu,\nu}(x) = O(1)$ for x small. Hence (4.2) follows.

Sufficiency: Since (4.2) can be written as

$$\frac{\Gamma(\mu+\frac{1}{2}-it)\Gamma(\nu+\frac{1}{2}-it)}{\Gamma(\mu+\frac{1}{2}+it)\Gamma(\nu+\frac{1}{2}+it)}\mathfrak{R}(t)\mathfrak{M}_t f = \frac{\Gamma(\xi+\frac{1}{2}+it)\Gamma(\eta+\frac{1}{2}+it)}{\Gamma(\xi+\frac{1}{2}-it)\Gamma(\eta+\frac{1}{2}-it)}\mathfrak{Z}(-t)\mathfrak{M}_t f.$$

i.e., $$\left\{\int_0^\infty y^{-\frac{1}{2}-it}F(y)dy\right\}\mathfrak{Z}(t) = \frac{\Gamma(\xi+\frac{1}{2}+it)\,\Gamma(\eta+\frac{1}{2}+it)}{\Gamma(\xi+\frac{1}{2}-it)\,\Gamma(\eta+\frac{1}{2}-it)}\mathfrak{M}_{-t}\left\{\int_0^\infty k(xy)f(y)dy\right\}$$

i.e., $$\left\{\int_0^\infty x^{-\frac{1}{2}+it}dx\int_0^\infty K(xy)F(y)dy = \int_0^\infty x^{-\frac{1}{2}+it}dx\left\{\mathfrak{S}_{2\xi,\,2\eta}\int_0^\infty k(xy)f(y)\,dy\right\}\right.$$

and hence the rule follows.

The proof of the Rule 2 is quite similar.

The representations

$$\begin{matrix}K(x)\\k(x)\end{matrix} = \mathfrak{M}_x^{-1}\left[\Gamma(\mu+\tfrac{1}{2}+it)\Gamma(\nu+\tfrac{1}{2}+it)\Gamma(\xi+\tfrac{1}{2}+it)\Gamma(\eta+\tfrac{1}{2}+it)\begin{matrix}\mathfrak{Z}(t)\\\mathfrak{Z}(-t)\end{matrix}\right] \tag{4.6}$$

of a pair of kernels for Rule 1 and

$$\begin{matrix}K(x)\\k(x)\end{matrix} = \mathfrak{M}_x^{-1}\left[\Gamma(\mu+\tfrac{1}{2}-it)\Gamma(\nu+\tfrac{1}{2}-it)\Gamma(\xi+\tfrac{1}{2}+it)\Gamma(\eta+\tfrac{1}{2}+it)\begin{matrix}\mathfrak{Z}(t)\\\mathfrak{Z}(-t)\end{matrix}\right] \tag{4.7}$$

of a pair of kernels for Rule 2 follow from (4.2) and (4.4) respectively. $\mathfrak{Z}(t)$ is an arbitrary function such that right hand sides of (4.6) and (4.7) respectively exist.

Some representations of the kernel: Let us consider

$$\lambda(x) = (1/2\pi)\int_{-\infty}^{\infty}\Gamma(\mu+\tfrac{1}{2}-it)\Gamma(\nu+\tfrac{1}{2}-it)\Gamma(\xi+\tfrac{1}{2}-it)\Gamma(\eta+\tfrac{1}{2}-it)x^{-\frac{1}{2}-it}dt \tag{4.8}$$

where we assume that $\mathbf{Re}(\mu-\frac{1}{2})$, $\mathbf{Re}(\nu-\frac{1}{2})$, $\mathbf{Re}(\xi-\frac{1}{2})$, $\mathbf{Re}(\eta-\frac{1}{2})$ are not negative integers.

Evaluating $\lambda(x)$ by considering the sum of the residues, we find

$$\lambda(x) = \Gamma(\mu+\xi+1)\Gamma(\nu+\xi+1)\Gamma(\eta-\xi)x^{\xi}\,{}_2F_1(\mu+\xi+1, \nu+\xi+1;\ 1+\xi-\eta;\ x) + \Gamma(\mu+\eta+1)\Gamma(\nu+\eta+1)\Gamma(\xi-\eta)x^{\eta}\,{}_2F_1(\mu+\eta+1, \nu+\eta+1;\ 1+\eta-\xi;\ x) \tag{4.9}$$

where $|x|<1$.

Let us next consider

$$\lambda_1(x) = (1/2\pi)\int_{-\infty}^{\infty}\Gamma(\mu+\tfrac{1}{2}+it)\Gamma(\nu+\tfrac{1}{2}+it)\Gamma(\xi+\tfrac{1}{2}+it)\Gamma(\eta+\tfrac{1}{2}+it)x^{-\frac{1}{2}-it}dt \tag{4.10}$$

where $\mathbf{Re}(\mu-\frac{1}{2})$, $\mathbf{Re}(\nu-\frac{1}{2})$, $\mathbf{Re}(\xi-\frac{1}{2})$, $\mathbf{Re}(\eta-\frac{1}{2})$ are not negative integers. Evaluating as the some of the residues, we find

$$\lambda_1(x)=\Gamma(\nu-\mu)\Gamma(\xi-\mu)\Gamma(\eta-\mu)x^{\mu}\,{}_0F_3(\,;1+\mu-\nu, 1+\mu-\eta, 1-\mu-\xi;-x) + \mathrm{idem}(\mu;\nu,\xi,\eta) \tag{4.11}$$

where idem $(a\,;b)$ after an expression means the proceeding expression with a and b interchanged.

Hence the representations

$$\begin{matrix}K(x)\\k(x)\end{matrix} = \int_0^\infty \lambda_1(x)\begin{matrix}h(y)dy\\y^{-1}h(y^{-1})dy\end{matrix} \tag{4.12}$$

and

$$\begin{matrix}K(x)\\k(x)\end{matrix} = \int_0^\infty \lambda(x)\begin{matrix}h(y)dy\\y^{-1}h(y^{-1})dy\end{matrix} \tag{4.13}$$

for the rules 1 and 2. In both (4.12) and (4.13) $h(y)$ is an arbitrary function of $L_2(0, \infty)$.

5. We now consider an operator $T_{\eta,\alpha}$ dafined as (Erdélyi, 1941)

$$T_{\eta,\alpha}f(x) = \underset{\delta\to 0}{\mathrm{l.i.m.}}\ x^{-\frac{1}{2}\alpha}\int_\delta^\infty y^{\frac{1}{2}\alpha-1}J_{2\eta+\alpha}(2\sqrt{x/y})\,f(y)\,dy \tag{5.1}$$

where $\mathbf{Re}\,\eta>-\frac{1}{2}$. The integral is convergent in the ordinary sense if $\mathbf{Re}\,\alpha>\frac{1}{2}$. It is easy to see that

$$\mathfrak{M}_t(T_{\eta,\alpha}f) = \frac{\Gamma(\eta+\frac{1}{2}+it)}{\Gamma(\alpha+\eta+\frac{1}{2}-it)}\mathfrak{m}_t f \tag{5.2}$$

Formulae (4.2) and (4.4) suggest that

either
$$\begin{matrix}K(x)\\k(x)\end{matrix} = T_{\mu,\xi-\mu}\left\{T_{\nu,\eta-\nu}\begin{pmatrix}h(x)\\x^{-1}h(x^{-1})\end{pmatrix}\right\} \tag{5.3}$$

or
$$\begin{matrix}K(x)\\k(x)\end{matrix} = T_{\xi,\mu-\xi}\left\{T_{\eta,\nu-\eta}\begin{pmatrix}h(x)\\x^{-1}h(x^{-1})\end{pmatrix}\right\}$$

should be a suitable pair of kernels for Rule 1, and that

either
$$\begin{matrix}K(x)\\k(x)\end{matrix} = I^{+}_{\mu,\xi-\mu}\left\{I^{+}_{\nu,\eta-\nu}\begin{pmatrix}h(x)\\h^{-1}h(x^{-1})\end{pmatrix}\right\}$$

or

$$\begin{matrix} K(x) \\ k(x) \end{matrix} = K_{\bar{\xi},\,\mu-\xi}\left\{K_{\eta,\,\nu-\eta}\begin{pmatrix} h(x) \\ h^{-1}h(x^{-1}) \end{pmatrix}\right\} \tag{5.4}$$

should be a suitable pair of kernels for Rule 2. In both (5.3) and (5.4) $h(x)$ is an arbitrary function such that (5.8) and (5.4) have a meaning. The truth of (5.3) and (5.4) can be verified by taking the Mellin transforms of both sides.

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STRESSES DUE TO A CONCENTRATED FORCE IN AN INFINITE ELASTIC PARABOLIC PLATE NOT CONTAINING THE FOCUS

By

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(*Communicated by Dr. D. N. Mitra—Received August 7, 1957*)

Summary : In this paper the method of complex function theory has been employed to solve a two dimensional problem for an elastic plate with stress free parabolic boundary not containing the focus under the effect of a concentrated force acting at any point of the plate when the stresses and rotation vanish at infinity. The method leads to the solution of Hilbert problem.

1. The first boundary value problem of plane theory of elasticity may be solved by Muskhelishvili's method of complex function theory when there are no singularities inside the elastic material. The problem when concentrated force and couple act in the interior of the region has been solved in some cases by Green and Zerna (1954). Here we work out the problem for the plate with a parabolic boundary not containing the focus when a force acts at any point in its plane, the boundary being supposed stress free.

The elastic region is mapped conformally to the half plane, the parabolic boundary transforming to straight boundary. Using the principle of analytic continuation the problem is reduced to Muskhelishvili's so-called problem of linear relationship for the determination of a single sectionally holomorphic function. The particular case when the force acts at the vertex of the parabola is obtained from the general solution and the result in the case of normal force at the vertex is seen to agree with that derived from Tiffen's (1952) solution of uniform distribution of normal pressure over a finite arc, symmetrical about the vertex, of the parabolic boundary.

2. The stress components at any point of the z-plane $(z = x+iy)$ are expressible in terms of two complex potential functions $\varphi(z)$ and $\psi(z)$, (using the notation of Green and Zerna, 1954), as

$$\left.\begin{aligned} \sigma_{xx}+\sigma_{yy} &= 4[\varphi(z)+\bar{\varphi}'(\bar{z})] \\ \sigma_{xx}-\sigma_{yy}+2i\tau_{xy} &= -4[z\bar{\varphi}''(\bar{z})+\bar{\psi}'(\bar{z})] \end{aligned}\right\} \tag{2.1}$$

where the dash over a function represents derivative with respect to its argument

Suppose there acts a force $P = X+iY$ at a finite point $z = z_0$ of the parabolic plate. For the unique solution (Green and Zerna 1954) the functions $\varphi'(z)$, $\psi'(z)$ must, for large values of $|z|$, satisfy the condition

$$\left.\begin{aligned} \varphi'(z) &= -\frac{X+iY}{8\pi z}+O\left(\frac{1}{z}\right), \\ \psi'(z) &= \frac{X-iY}{8\pi z}+O\left(\frac{1}{z}\right) \end{aligned}\right\} \tag{2.2}$$

provided the stresses and rotation vanish at infinity.

The potentials $\varphi_0(z)$, $\psi_0(z)$ which define a stress system due to the concentrated force are given by

$$\left.\begin{aligned} \varphi_0(z) &= -\frac{X+iY}{4\pi(k+1)}\log(z-z_0), \\ \psi_0(z) &= \frac{k(X-iY)}{4\pi(k+1)}\log(z-z_0)+\frac{X+iY}{4\pi(k+1)}\cdot\frac{\bar{z}_0}{z-z_0} \end{aligned}\right\} \tag{2.3}$$

where for plane strain

$$k = 3-4\sigma, \tag{2.4}$$

σ being Poisson's ratio.

We have to find complex potentials $\varphi_1(z)$, $\psi_1(z)$ so that the combined potentials

$$\varphi(z) = \varphi_0(z)+\varphi_1(z),\ \psi(z) = \psi_0(z)+\psi_1(z) \tag{2.5}$$

given zero applied stresses on the boundary and satisfy (2.2).

It is clear from (2.2), (2.3) that $\varphi'_1(z)$, $\psi'_1(z)$ must be $O(1/z)$ for large values of $|z|$.

3. The transformation formula

$$z = z(\zeta) = \frac{1}{2i}(\zeta+ia)^2,\ \zeta = \xi+i\eta,\ a>0 \tag{3.1}$$

maps conformally the exterior region *i.e.* one not containing the focus of the parabola

$$x^2 = -2a^2y+a^4 \tag{3.2}$$

to the upper half, $\eta>0$ of the ζ-plane. Let S^+, S^- denote the regions $\eta > 0$, $\eta<0$ respectively and let L represent $\eta = 0$ which corresponds to the parabolic boundary.

The stresses $\sigma_{\xi\xi}$, $\sigma_{\eta\eta}$, $\tau_{\xi\eta}$ in (ξ, η) coordinates can be expressed in terms of complex potentials $\varphi(\zeta)$, $\psi(\zeta)$ where

$$\varphi(z) = \varphi[z(\zeta)]\equiv\varphi(\zeta),\ \psi(z)\equiv\psi(\zeta). \tag{3.3}$$

and

$$\varphi(\zeta) = \varphi_0(\zeta)+\varphi_1(\zeta),\ \psi(\zeta) = \psi_0(\zeta)+\psi_1(\zeta). \tag{3.4}$$

However the formula (Muskhelishvili, 1953) expressing the stresses on the boundary can be easily written with the help of (2.1)

$$\sigma_{\eta\eta}-i\tau_{\xi\eta} = 2\left[\frac{\varphi'(\zeta)}{z'(\zeta)}+\frac{\bar{\varphi}'(\bar{\zeta})}{\bar{z}'(\bar{\zeta})}+\frac{z(\zeta)}{z'(\zeta)}+\frac{z(\zeta)}{z'(\zeta)}\frac{d}{d\bar{\zeta}}\frac{\bar{\varphi}'(\bar{\zeta})}{\bar{z}'(\bar{\zeta})}+\frac{\bar{\psi}'(\bar{\zeta})}{z'(\zeta)}\right]. \tag{3.5}$$

The functions $\frac{\varphi_1'(\zeta)}{z'(\zeta)}, \frac{\psi_1'(\zeta)}{z'(\zeta)}$ are defined in S^+. We extend the definition of $\frac{\varphi_1(\zeta)}{z(\zeta)}$ to the whole of ζ-plane by putting (for ζ in S^-)

$$\frac{\varphi_1'(\zeta)}{z'(\zeta)} = -\frac{\bar{\varphi}_1'(\zeta)}{\bar{z}'(\zeta)} - \frac{z(\zeta)}{z'(\zeta)}\frac{d}{d\zeta}\frac{\bar{\varphi}_1'(\zeta)}{\bar{z}'(\zeta)} - \frac{\bar{\psi}_1'(\zeta)}{z'(\zeta)} \tag{3.6}$$

If ζ belong to S^+, $\bar{\zeta}$ belongs to S^-, so that from (3.6) we get

$$\frac{\varphi_1'(\bar{\zeta})}{z'(\bar{\zeta})} = -\frac{\bar{\varphi}_1'(\bar{\zeta})}{\bar{z}'(\bar{\zeta})} - \frac{z(\bar{\zeta})}{z'(\bar{\zeta})}\frac{d}{d\bar{\zeta}}\frac{\bar{\varphi}_1'(\bar{\zeta})}{\bar{z}'(\bar{\zeta})} - \frac{\bar{\psi}_1'(\bar{\zeta})}{z'(\bar{\zeta})}$$

or

$$\psi_1'(\zeta) = -\bar{\varphi}_1'(\zeta) - \bar{z}'(\zeta)\frac{\varphi_1'(\zeta)}{z'(\zeta)} - \bar{z}(\zeta)\frac{d}{d\zeta}\frac{\varphi_1'(\zeta)}{z'(\zeta)} (\zeta \text{ in } S^+) \tag{3.7}$$

With the help of relation (3.6), the formula (3.5) becomes for ζ in S^+

$$\tfrac{1}{2}(\sigma_{\eta\eta} - i\tau_{\xi\eta}) = \frac{\varphi_1'(\zeta)}{z'(\zeta)} - \frac{\varphi_1'(\bar{\zeta})}{z'(\bar{\zeta})} + \left(1 - \frac{z'(\bar{\zeta})}{z'(\zeta)}\right)\frac{\bar{\varphi}_1'(\bar{\zeta})}{\bar{z}'(\bar{\zeta})} + \left(\frac{z(\zeta)}{z'(\zeta)} - \frac{z(\bar{\zeta})}{z'(\bar{\zeta})}\right)\frac{d}{d\bar{\zeta}}\frac{\bar{\varphi}_1'(\bar{\zeta})}{\bar{z}'(\bar{\zeta})}$$

$$+\frac{\varphi_0'(\zeta)}{z'(\zeta)} + \frac{\bar{\varphi}_0'(\bar{\zeta})}{\bar{z}'(\bar{\zeta})} + \frac{z(\zeta)}{z'(\zeta)}\frac{d}{d\bar{\zeta}}\frac{\bar{\varphi}_0'(\bar{\zeta})}{\bar{z}'(\bar{\zeta})} + \frac{\psi_0'(\bar{\zeta})}{z'(\zeta)}. \tag{3.8}$$

We assume that as η tends to zero, the third and fourth terms on the right hand side vanish identically and that limits $\left[\frac{\varphi_1'(t)}{z'(t)}\right]^+$ and $\left[\frac{\varphi_1'(t)}{z'(t)}\right]^-$ exist for all t on L. If the applied stress on the boundary vanish, we get (for t on L)

$$\left[\frac{\varphi_1'(t)}{z'(t)}\right]^+ - \left[\frac{\varphi_1'(t)}{z'(t)}\right]^- = -\frac{\varphi_0'(t)}{z'(t)} - \frac{\bar{\varphi}_0'(t)}{\bar{z}'(t)} - \frac{z(t)}{z'(t)} \cdot \frac{d}{dt}\frac{\bar{\varphi}_0'(t)}{\bar{z}'(t)} - \frac{\bar{\psi}_0'(t)}{z'(t)}. \tag{3.9}$$

The right hand side of (3.9) expresses the discontinuity of $\frac{\varphi_1'(\zeta)}{z'(\zeta)}$ on the boundary L.

From the definition of complex potentials $\frac{\varphi_0'(\zeta)}{z'(\zeta)}, \frac{\psi_0'(\zeta)}{z'(\zeta)}$ we observe that right hand side satisfies Hölder's condition and thus the unique solution $\varphi_1'(\zeta)/z'(\zeta)$, sectionally holomorphic everywhere except at the point $\zeta = -ia$ and which vanishes at infinity, satisfying the boundary condition (3.9) is given by

$$\frac{\varphi_1'(\zeta)}{z'(\zeta)} = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\left(\frac{\varphi_0'(t)}{z'(t)} + \frac{\bar{\varphi}_0'(t)}{\bar{z}'(t)} + \frac{z(t)}{z'(t)}\frac{d}{dt}\frac{\bar{\varphi}_0'(t)}{z'(t)} + \frac{\bar{\psi}_0'(t)}{z'(t)}\right)\frac{dt}{t-\zeta} + \frac{A}{\zeta + ia} \tag{3.10}$$

where A is an unknown constant.

4. The integrals in (3.10) can be easily evaluated by theory of residues. Thus we obtain, (for ζ in S^+)

$$\frac{\varphi_1'(\zeta)}{z'(\zeta)} = \frac{Y-iX}{4\pi(k+1)}\cdot\frac{1}{(\zeta_0+ia)(\zeta+\zeta_0+2ia)} - \frac{Y+iX}{4\pi(k+1)}\,\frac{1}{(\bar{\zeta}_0-ia)(\zeta-\bar{\zeta}_0)} + \frac{Y+iX}{8\pi(k+1)}\,\frac{(\bar{\zeta}_0-ia)(\zeta+ia)}{(\zeta-\bar{\zeta}_0)^2(\bar{\zeta}_0-ia)^2}$$

$$+\frac{Y+iX}{8\pi(k+1)}\left\{\frac{(\zeta_0+ia)^2(\zeta+2\bar{\zeta}_0-5ia)}{(\bar{\zeta}_0-3ia)^2(\zeta+\bar{\zeta}_0-2ia)^2(\bar{\zeta}_0-ia)} - \frac{4(\zeta_0+ia)^2(\zeta-ia)}{(\zeta-\bar{\zeta}_0)^2(\zeta+\bar{\zeta}_0-2ia)^2(\zeta+ia)}\right\}$$

$$+\frac{k(Y-iX)}{4\pi(k+1)}\cdot\frac{\zeta-3ia}{(\zeta+ia)(\zeta-\bar{\zeta}_0)(\bar{\zeta}_0-3ia)} + \frac{A}{\zeta+ia} \tag{4.1}$$

whereas (for ζ in S^-)

$$\frac{\varphi_1'(\zeta)}{z'(\zeta)} = \frac{Y-iX}{4\pi(k+1)}\cdot\frac{1}{(\zeta_0+ia)(\zeta-\zeta_0)} - \frac{Y+iX}{4\pi(k+1)}\,\frac{1}{(\bar{\zeta}_0-ia)(\zeta+\bar{\zeta}_0-2ia)} + \frac{Y+iX}{8\pi(k+1)} \times$$

$$\frac{(\zeta_0+ia)^2(\zeta+2\bar{\zeta}_0-5ia)}{(\bar{\zeta}_0-3ia)^2(\bar{\zeta}_0-ia)(\zeta+\bar{\zeta}_0-2ia)^2} + \frac{Y+iX}{8\pi(k+1)}\cdot\frac{2(\bar{\zeta}-2ia)(\zeta+\bar{\zeta}_0-2ia)-(\bar{\zeta}_0-3ia)(\zeta+2\bar{\zeta}_0-3ia)}{(\bar{\zeta}_0-ia)^2(\zeta+\bar{\zeta}_0-2ia)^2}$$

$$+\frac{k(Y-iX)}{4\pi(k+1)}\cdot\frac{1}{(\bar{\zeta}_0-3ia)(\zeta+\bar{\zeta}_0-2ia)} + \frac{A}{\zeta+ia} \tag{4.2}$$

where ζ_0 corresponds to the point z_0.

With the help of (2.2), (2.3), (2.5), (3.3), it is verified that $\varphi_1'(\zeta)/z'(\zeta)$ is $O(1/\zeta^2)$ for large $|\zeta|$ and hence the unknown constant A is determined by making zero the coefficient of $1/\zeta$ in the expansion of $\varphi_1'(\zeta)/z'(\zeta)$ for large $|\zeta|$.

Thus

$$A = -\frac{Y-iX}{4\pi(k+1)}\,\frac{1}{\zeta_0+ia} + \frac{Y+iX}{8\pi(k+1)}\,\frac{1}{\bar{\zeta}_0-ia} - \frac{k(Y-iX)}{4\pi(k+1)}\cdot\frac{1}{\bar{\zeta}_0-3ia} - \frac{Y+iX}{8\pi(k+1)}\cdot\frac{(\zeta_0+ia)^2}{(\bar{\zeta}_0-3ia)^2(\bar{\zeta}_0-ia)}.$$

With the above value of A, (4.1) and (4.2) simplify to

$$\frac{\varphi_1'(\zeta)}{z'(\zeta)} = -\frac{Y-iX}{4\pi(k+1)}\cdot\frac{1}{(\zeta+ia)(\zeta+\zeta_0+2ia)} + \frac{k(Y-iX)}{4\pi(k+1)}\cdot\frac{1}{(\zeta-\bar{\zeta}_0(\zeta+ia)}$$

$$+\frac{Y+iX}{8\pi(k+1)}\cdot\frac{(\bar{\zeta}_0+ia)^2-(\zeta_0+ia}{(\zeta+ia)(\bar{\zeta}_0-ia)(\zeta-\bar{\zeta}_0)^2} \qquad (\zeta \text{ in } S^+), \tag{4.4}$$

$$\frac{\varphi_1'(\zeta)}{z'(\zeta)} = \frac{Y-iX}{4\pi(k+1)}\cdot\frac{1}{(\zeta-\zeta_0)(\zeta+ia)} - \frac{k(Y-iX)}{4\pi(k+1)}\cdot\frac{1}{(\zeta+ia)(\zeta+\zeta_0-2ia)}$$

$$+\frac{Y+iX}{8\pi(k+1)}\cdot\frac{(\bar{\zeta}_0-3ia)^2-(\zeta_0+ia)^2}{(\zeta+ia)(\bar{\zeta}_0-ia)(\zeta+\bar{\zeta}_0-2ia)^2} \qquad (\zeta \text{ in } S^-). \tag{4.5}$$

Hence the potentials defining the elastic state due to concented force $X+iY$ in a parabolic plate with stress-free boundary, are

$$\frac{\varphi'(\zeta)}{z'(\zeta)} = \frac{\varphi_0'(\zeta)}{z'(\zeta)}+\frac{\varphi_1'(\zeta)}{z'(\zeta)} = \frac{Y-iX}{4\pi(k+1)}\left[\frac{1}{(\zeta+ia)(\zeta-\zeta_0)}+\frac{k}{(\zeta+ia)(\zeta-\bar{\zeta}_0)}\right].$$

$$+\frac{Y+iX}{8\pi(k+1)}\cdot\frac{(\bar{\zeta}_0+ia)^2-(\zeta_0+ia)^2}{(\zeta+ia)(\bar{\zeta}_0-ia)(\zeta-\bar{\zeta}_0)^2} \qquad (4.6)$$

and
$$\frac{\psi'(\zeta)}{z'(\zeta)} = \frac{\psi_0'(\zeta)}{z'(\zeta)}+\frac{\psi_1'(\zeta)}{z'(\zeta)} = \frac{\psi_0'(\zeta)}{z'(\zeta)}-\frac{\bar{z}'(\zeta)}{z'(\zeta)}\left[\frac{\varphi_1'(\zeta)}{z'(\zeta)}+\frac{\bar{\varphi}_1'(\zeta)}{\bar{z}'(\zeta)}\right]-\frac{\bar{z}(\zeta)}{z'(\zeta)}\frac{d}{d\zeta}\frac{\varphi_1'(\zeta)}{z'(\zeta)} \qquad (4.7)$$

where
$$\frac{\psi_0'(\zeta)}{z'(\zeta)} = \frac{k(Y+iX)}{2\pi(k+1)}\cdot\frac{1}{(\zeta-\zeta_0)(\zeta+\zeta_0+2ia)}-\frac{Y-iX}{2\pi(k+1)}\cdot\frac{(\zeta_0-ia)^2}{(\zeta-\zeta_0)^2(\zeta+\zeta_0+2ia)^2} \qquad (4.8)$$

and the full expression for this complex potential can be written using (4.4) and (4.5).

It is easily seen from (4.6), (4.7) that for large $|\zeta|$

$$\left.\begin{aligned}\frac{\varphi'(\zeta)}{z'(\zeta)} &= \frac{Y-iX}{4\pi\zeta^2}+0\left(\frac{1}{\zeta^3}\right)\\ \text{and}\qquad \frac{\psi'(\zeta)}{z'(\zeta)} &= \frac{Y+iX}{4\pi\zeta^2}+0\left(\frac{1}{\zeta^3}\right)\end{aligned}\right\} \qquad (4.9)$$

which are the correct forms as can be verified using (2.2). The complete distribution of stress and displacement may now be obtained from complex potentials (4.6), (4.7). However the value of $\sigma_{\eta\eta}$ on the boundary is easily evaluated.

Let $\zeta = ib$, $b>0$ *i.e.* z_0 is a point on the axis of the parabola.

$$\sigma_{\xi\xi} = \frac{2}{\pi(\xi^2+a^2)(\xi^2+b^2)}\left\{\xi X\left[\frac{b(1-k)}{1+k}-a\right]+Y\left[\frac{ab(1-k)}{1+k}+\xi^2\right]\right\}$$

$$+\frac{4ab}{\pi(a+b)(\xi^2+a^2)(\xi^2+b^2)^2}\{\xi X(b^2+2ab-\xi^2)+Y(2b\xi^2+a\xi^2-ab^2)\} \qquad (4.10)$$

5. Concentrated Force at the Vertex of the Parabola. It is seen from the above solution that we can derive the effect of a force $P = X+iY$ acting at the vertex of the parabola, which corresponds to the origin of the ζ-plane. The complex potentials (4.6), (4.7) reduce to simple expressions

$$\frac{\varphi'(\zeta)}{z'(\zeta)} = \frac{Y-iX}{4\pi\zeta(\zeta+ia)}, \qquad (5.1)$$

$$\frac{\psi'(\zeta)}{z'(\zeta)} = \frac{Y+iX}{4\pi\zeta(\zeta+ia)}+\frac{a(Y-iX)(\zeta-ia)(3i\zeta-a)}{8\pi\zeta^2(\zeta+ia)^3} \qquad (5.2)$$

The stress distribution is completely determined from (5.1) (5.2) and the value of $\sigma_{\xi\xi}$ on the boundary is given by

$$\sigma_{\xi\xi} = \frac{2(\xi Y - aX)}{\pi\xi(a^2+\xi^2)} . \tag{5.3}$$

6. The solution given by (5 1), (5.2) can also be derived from the problem of uniform distribution of pressure over a finite arc l, symmetrical about the vertex of the boundary of the parabola. A similar problem when there acts normal pressure on the arc of the parabolic boundary has been worked by Tiffen (1952) using a combination of complex potential and Fourier Integral method.

Let the uniform pressure, p_n along the normal and p_t along the tangent, act at every point of the boundary arc l and let $\xi = -b$, $\xi = b$ correspond to end points of it. By making $\varphi_0(\zeta) \equiv 0$, $\varphi_1(\zeta) \equiv \varphi(\zeta)$ in formulae of para. **3**, we get the boundary condition from formula (3.8)

$$\left[\frac{\varphi'(t)}{z'(t)}\right]^+ - \left[\frac{\varphi'(t)}{z'(t)}\right]^- = -\frac{p}{2} \text{ (for } t \text{ on } l) \tag{6.1}$$

where

$$p = p_n - ip_t \tag{6.2}$$

The unique solution of this boundary condition is given by

$$\frac{\varphi'(\zeta)}{z'(\zeta)} = -\frac{p}{4\pi i} \log\frac{\zeta-b}{\zeta+b} - \frac{bp}{2\pi i} \cdot \frac{1}{\zeta+ia} \tag{6.3}$$

and on using (6.3), formula (3.7) gives

$$\frac{\psi'(\zeta)}{z'(\zeta)} = \frac{\bar{p}-p}{4\pi i} \cdot \frac{\zeta-ia}{\zeta+ia} \log\frac{\zeta-b}{\zeta+b} + \frac{b\bar{p}}{2\pi i} \cdot \frac{1}{\zeta+ia} - \frac{bp}{2\pi i} \cdot \frac{\zeta-ia}{(\zeta+ia)^2}$$

$$- \frac{bp}{4\pi i} \cdot \frac{(\zeta-ia)^2}{(\zeta+ia)(\zeta^2-b^2)} + \frac{bp}{4\pi i} \cdot \frac{(\zeta-ia)^2}{(\zeta+ia)^3} \tag{6.4}$$

If we put $p = 1$ the potentials (6 3), (6 4) reduce to those obtained in Tiffen's paper from relations (68) by using relations (40) and (63), (43) and (63) respectively of that paper in the case of uniform normal distribution of pressure. However it may be mentioned that instead of relation (68) of Tiffen's paper we have

$$w_0(\zeta) = \frac{i\zeta}{6}(\zeta^2+6ia\zeta+3a^2) \text{ and not } \frac{i\zeta}{6}(\zeta^2+15a^2).$$

The total resultant force $-P$ acting over l is

$$-P = \int_l (X_n+iY_n)ds = \int_{-b}^{b} (\tau_{\xi\eta}+i\sigma_{\eta\eta}) \frac{z'(t)}{|z'(t)|} |z'(t)| \, dt = -2iabp. \tag{6.5}$$

Now if $P = X + iY$ is the concentrated force at the vertex, then proceeding to the limiting case when b tends to zero but $2iabp$ remains finite and equal to P we get from (6.3), (6.4)

$$\frac{\varphi'(\zeta)}{z'(\zeta)} = \frac{-iP}{4\pi\zeta(\zeta+ia)}, \tag{6.6}$$

$$\frac{\psi'(\zeta)}{z'(\zeta)} = \frac{i\bar{P}}{4\pi\zeta(\zeta+ia)} - \frac{iaP}{8\pi}\frac{(\zeta-ia)(3i\zeta-a)}{\zeta^2(\zeta+ia)^3}. \tag{6.7}$$

The complex potentials (6.6), (6.7) are same as (5.1), (5.2) respectively and these completely solve the problem of the concentrated force at the vertex of the parabola.

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ON A QUESTION OF REPRESENTATION OF ANALYTIC FUNCTIONS BY FABER POLYNOMIALS

By

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In this paper we consider a question, suggested by Lokhin (1955), on the representation of regular analytic functions of a complex variable by means of Faber polynomials. The question is: What is the representation of such functions inside and outside their regularity domain?

1. Let G be a domain of finite connectivity with contours which do not reduce to a point. We denote by $\overline{G}$ the closure and by G^* the complement of this domain. Consider a class of analytic and schlicht functions

$$w = w(z) = z\sum_{\nu=0}^{\infty} a_\nu z^{-\nu}, \ a_0 = 1, \tag{1}$$

where $w(\infty)$ and $w'(\infty) > 0$, which map $\overline{G}$ conformally and one-to-one onto the unit circle $|w| \leq 1$ and G^* onto the exterior of this unit circle, so that the points at infinity in the two planes correspond to each other.

We assume that the contours $C_{r_i} (i = 0, 1, 2, \ldots)$ are closed analytic curves; the complete boundary of G will be denoted by C. Further, let C_{r_0} denote that closed analytic curve which is mapped by the functions (1) onto the circumference of the circle $|w| = r_0 < 1$.

We understand by the Faber polynomials $w_n(z)$ of the n-th degree the totality of magnitudes with nonnegative powers of z in the Laurent series of $w^n(z)$ in the neighbourhood of the point $z = \infty$[see Lokhin (1955), Schiffer (1848), Schur (1945) and Szego (1939)]. The existence and uniqueness of all Faber polynomials $w_n(z)$ for $n \geq 1$ is easily shown by recursion.

Now any regular analytic function $f(z)$ may be represented in $\overline{G}$ by a series with the Faber polynomials as (cf. Lokhin, 1955)

$$f(z) = \sum_{n=1}^{\infty} A_n w_n(z), \tag{2}$$

where

$$A_n = \frac{1}{2\pi i} \int f[z(w)] w^{-n-1}\, dw, \ n = 0, 1, 2, \ldots\ldots, \tag{3}$$

$z(w)$ being the inverse function of $w = w(z)$, and the integration extended over a closed analytic curve around the origin.

The outline of the solution is as follows: If C_{r_1}, $r_0 \leq r_1$ be the first of the closed analytic curves, in whose closure $f(z)$ is regular analytic, then the series (2) will converge inside and diverge outside C_{r_1}. But if $f(z)$ could be continued analytically across C_{r_1}, then naturally the question of summation of the series (2) outside C_{r_2} arises. We have followed the method suggested by Dienes (1931), which is based on the analytic continuation of $f(z)$ to Mittag-Leffler's star domain.

Let $D\subset G$ be the regularity domain of $f(z)$. If we denote by d the image domain, onto which D is mapped conformally by (1), then d will contain the unit circle $|w|\leq 1$ inside itself. We now consider in D three closed analytic curves C_{r_1}, C_{r_2} and C_{r_3}, such that $C_{r_1}\subset C_{r_2}\subset C_{r_3}$. Let the images of these analytic curves be denoted by Γ_1, Γ_2 and Γ_3, respectively. We notice that $f(z)$ is regular analytic on these curves and between them.

We shall at first establish a formula to represent $f(z)$ inside D by means of an auxiliary summatory function φ, which defines the coefficients of the series (2).

Theorem 1. *The regular analytic function $f(z)$ may be uniquely represented inside its regularity domain D by means of an auxiliary summatory function $\varphi(t)$, $t = w(\zeta)$, as*

$$f(z) = \frac{1}{2\pi i}\int_{C_{r_1}\subset D}\frac{\varphi[w(\zeta)]}{\zeta - z}d\zeta, \quad z\in D, \tag{4}$$

where

$$\varphi(t) = \sum_{n=0}^{\infty} A_n t^n \tag{5}$$

and A_n are the coefficients (3).

Proof. The function

$$g(t) = \frac{1}{2\pi i}\int_{\Gamma_2}\frac{f[z(w)]}{w-t}dw$$

is regular analytic outside Γ_2, where $g(\infty) = 0$. If $t\in\Gamma_1$, then by Cauchy's theorem we have (cf. also Szegö, 1939)

$$g(t) = -f[z(t)] + \varphi(t), \tag{6}$$

where

$$\varphi(t) = \frac{1}{2\pi i}\int_{\Gamma_3\subset d}\frac{f[z(w)]}{w-t}dw. \tag{7}$$

Put $z(t) = \zeta$, $t = w(\zeta)$ in (6). Then

$$\varphi[w(\zeta)] = f(\zeta) + g[w(\zeta)]. \tag{8}$$

Thus $g[w(\zeta)]$ is regular analytic outside C_{r_1}, where $g[w(\infty)] = 0$. Therefore

$$\frac{1}{2\pi i}\int_{C_{r_1}}\frac{g[w(\zeta)]}{\zeta - z}\,d\zeta = 0 \tag{9}$$

for all values of $z\in C_{r_1}\subset D$ (see also Dienes, 1931, p. 309). Now if we multiply (8) by $(2\pi i)^{-1}(\zeta - z)^{-1}$ and integrate over C_{r_1}, then in view of the relation (9) we get the representation (4).

Since the choice of the closed analytic curve C_{r_1} is arbitrary, the formula (4) represents inside D the regular analytic function $f(z)$ by means of the auxiliary summatory function $\varphi(t)$. The representation (5) of $\varphi(t)$ follows directly from (7) in view of the theorem itself. Further, the uniqueness of this representation of $f(z)$ inside D follows obviously from (4) and (5).

2. We shall now show that the relations (4) and (5) make the summation of the series (2) outside D possible. For this purpose we introduce the Mittag-Leffler's star function with respect to G. According to Dienes (1931, p. 308) the analytic continuation of $f(z) = \sum A_n w_n(z)$ across the boundary of D gives rise to the star domain M with respect to the origin of coordinates of G.

Thus we prove for the representation of $f(z)$ outside D:

Theorem 2. *In the star domain M the following representation of the regular analytic function $f(z)$ holds:*

$$f(z) = \lim_{n\to\infty} \sum_{\nu=0}^{n} A_\nu k_\nu w_\nu(z) \tag{10}$$

where A_ν are the coefficients in the series (2), $w_\nu(z)$ the Faber polynomials and k_ν complex numbers independent of $f(z)$ and $w(z)$.

The righthand-side series in 10) is uniformly convergent in an arbitrary domain $D_0 \subset M$.

Proof. Let D_0 be contained inside C_{r_1} lying in M. Then the summatory function $\varphi(t)$ is regular analytic in M. Thus applying Mittag-Leffler's method of summation (see Dienes, p. 311) we obtain

$$\varphi(t) = \lim_{n\to\infty} \sum_{\nu=0}^{n} A_\nu k_\nu t^\nu, \tag{11}$$

where k_ν do not depend on $\varphi(t)$. This series will be uniformly convergent in any closed domain in M. Put $t = w(\zeta)$ in (11). Then

$$\varphi[w(\zeta)] = \lim_{n\to\infty} \sum_{\nu=0}^{n} A_\nu k_\nu [w(\zeta)]^\nu. \tag{12}$$

The series (12) is again uniformly convergent in any closed domain in M, in particular in the closure of C_{r_1}. Now if we multiply (12) by $(2\pi i)^{-1}(\zeta - z)^{-1}$, $z \in D_0$, and integrate over C_{r_1}, then in view of (4) we shall obtain (10), which proves the theorem completely.

Remarks. (1) The representation of $f(z)$ in this theorem can, however, be extended to the curvilinear star domain. But we omit such considerations here, as they may be easily persued according to Dienes (1931, p. 313).

(2) The representation (10) may also be justified as follows [cf. (Dienes, 1931, p. 311—2)]: Consider an arbitrary closed analytic curve C_0 in the z-plane, which contains the segment $[0, 1]$ inside itself. Then there is a conformal map $z = z(w)$, by which the domain lying inside C_0 is transformed into a circle of radius >1 in the w-plane such that $w = 0$ and $w = 1$ correspond to the points $z = 0$ and $z = 1$, respectively. Since $z(w)$ is regular at the origin and vanishes there, we have, for sufficiently small $|w|$ $|z(w)| < \varepsilon$ so small that we can substitute $z = z(w)$ in the Taylor's series

$$\varphi(z) = \sum_{n=0}^{\infty} A_n z^n.$$

Then

$$\varphi[z(w)] = \sum_{n=0}^{\infty} A_n [z(w)]^n.$$

But

$$z(w) = \sum_{n=1}^{\infty} A_n w^n, \tag{13}$$

and thus

$$\varphi[z(w)] = \sum_{n=0}^{\infty} P_n(A_n) w^n,$$

where $P_n(A_n)$ is a homogeneous linear expression in $A_0, A_1, \ldots, A_n$ satisfying the relation (3). If $\varphi(z)$ is regular inside C_0, $\varphi[z(w)]$ will be regular in the closure of the unit circle $|w| = 1$, *i.e.* the radius of convergence of the series (13) is > 1. We can apply the same syllogism to $f(z)$, represented by (10), whence the justification of that representation outside D follows.

(3) We repeat here a remark by Dienes (1931, p. 314): The only defect of all representations of the type (10) or (13) is that none of them converges in the whole star domain M but only in closed domains contained in M, which consequently tend to M if the arbitrary closed analytic curve C_{r_1} or C_0 is taken very close to the boundary of M.

We can, however, construct a representation of $f(z)$ in the complete star domain M by considering the absolute convergence of the series (10) in M. For this purpose let $s_n(z)$ denote the n-th partial sum of the series (10):

$$s_n(z) = \sum_{\nu=0}^{n} A_\nu k_\nu w_\nu(z).$$

Then for $|w| = 1$, $|f(z) - s_n(z)| < \sum_{n=1}^{\infty} \epsilon_n < \infty$, where ϵ_n depend on C_{r_1}. Next we choose η_n such that $|A_n - A_{n-1}| < \eta_n$ for $\eta_n < \epsilon_n$, where $\eta_n \to 0$ as $n \to \infty$. Under these considerations we obtain the representation of $f(z)$ outside D as*

$$f(z) = \lim_{n\to\infty} \sum_{\nu\to\infty}^{n} A_\lambda k_{n\nu} w_\nu(z), \tag{14}$$

where $k_{n\nu}$ depend only on ϵ_n, so that the series in (14) converges absolutely in the star domain M.

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* Lokhin (1955, p. 443) has also enunciated such a result, but his proof is incomplete in that he has not at all considered the absolute convergence of the series (14) in M.

A PROPERTY OF SETS OF POSITIVE MEASURE

By

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Kestelman (1947) has considered the problem when a convergent sequence "belongs" to a set. Given a set E of real numbers he asks: if $\{\lambda_n\}$ be a null sequence *i.e.*, $\lambda_n \to 0$ as $n \to \infty$ is there a number ξ such that $\xi + \lambda_n$ belongs to E for all large n? If there exists a ξ satisfying the above condition he says then that the sequence $\{\lambda_n\}$ belongs to E. If every null sequence belongs to E, then he says that E is universal and writes $E \epsilon U$. According to this definition, intervals are obviously universal sets. Kestelman (1947) has considered some sufficient conditions so that a set $E[\subset E_n$, where E_n is the Euclidean space of points or vectors $(x_1, x_2, \ldots x_n)$, x_n real] may be universal. He has shown that (1) if a set E contains almost all points of an interval, then $E \epsilon U$ (2) if a set E contains all the points of an interval except a set of the first category, then also $E \in U$. On the other hand, he has shown that if a set E is non-dense, then it cannot be universal.

Steinhaus (1920) has proved that if a set E is of positive measure, then the aggregate of the distances D of the points of E contains an interval whose left extremity is the point zero. Kestelman (1947) has obtained this result as a corollary of one of his theorems and if a set E satisfies this property, then he writes $E \epsilon S$. He has shown that the property of a set to be universal is stricter that the property of being an S-set *i.e.*, he has shown that if $E \epsilon U$ then $E \epsilon S$, but the converse is in general not true which he has verified by constructing an example C of a S-set for which C not ϵU.

F. W. Ghering (1951) has introduced the notion of "image" of a convergent sequence in a set in the following manner. Suppose that $\{x_n\}$ is any convergent sequence, then $\{x'_n\}$ is said to be an image of $\{x_n\}$, if there exists a positive number α such that $|x_n - x_m| = \alpha\,|x'_n - x'_m|$ for all m and n. ($|x-y|$ denotes the distance between x and y). From this it is easy to verify that if a set is universal, then it contains an image of any null sequence. On the other hand, Ghering (1951) has shown that if E is a non-dense set, then it is always possible to construct a convergent sequence which has no image in E.

In this article, we consider the following problem. Let α be a positive number and E be a set $\{\xi\}$ of vectors in the n-dimensional Euclidean Space E_n. By the symbol αE we mean the set of vectors $\{\alpha\xi\}$ in E_n.

Let $\alpha_1, \alpha_2, \ldots \alpha_k, \ldots$ be a sequence of real numbers such that $Lt\ \alpha_k = 1$ No generality will be lost if we take each $\alpha_k > 0$. Let E be any set of vectors in E_n. We enquire whether there exists a point $\xi \epsilon E_n$ such that the vectors $\xi/\alpha_1, \xi/\alpha_2, \ldots \xi/\alpha_k \ldots$ belong to E for all sufficiently large values of K. If such a ξ exists, we say that the sequence $\{\alpha_k\}$ permeates through the set E. If a set E be such that for every sequence $\{\alpha_k\}$, $\alpha_k > 0$, $Lt\ \alpha_k = 1$, there is a $\xi \epsilon E_n$ such that $\{\xi/\alpha_k\}$ belong to E for all large K, then we say that E is universally permeable.

According to this definition, intervals are obviously universally permeable sets and we have shown following the method adopted by Kestelman (1947) that if the set $S \subset I$ where I is an interval, then $I-S$ is universally permeable if (a) S is a set of measure zero, (b) S is a set of the first category.

The following theorems run parallel to the corresponding theorems of Kestelman (1947).

Theorem 1. *Let C be a closed set of positive measure contained in the open unit sphere in E_n and let p be any positive integer, then there is a positive number δ depending on p and the measure of C such that if*

$$\lambda_1, \lambda_2, \ldots \lambda_p$$

be any p real numbers such that

$$1 \leq \lambda_1 < \lambda_2 < \ldots < \lambda_p < 1+\delta$$

then the set of points $\xi \epsilon C$ such that

$$\xi/\lambda_1, \xi/\lambda_2, \ldots \xi/\lambda_p$$

also belong to C is closed and is of positive measure.

Proof. Let U be an open set containing C and itself contained within the unit sphere in E_n such that

$$|U|-|C| < |C|/p+1$$

Let $\delta\ (>0)$ be the distance between the closed sets C and U' [U' denotes the complement of u].

Let C_i denote the set $\lambda_i C (i = 1, 2, \ldots p)$. So C_i is the set of points $\{\lambda_i \eta\}$ where $\eta \epsilon C$. If $\xi \epsilon C$, then $\lambda_i \xi \epsilon C_i$ and $|\lambda_i \xi - \xi| = |\xi|(\lambda_i - 1) < |\xi| \delta < \delta$ since $|\xi| < 1$. So, $C_i \subset U$ for every $i = 1, 2, \ldots p$ and let us denote the set C by C_0.

Let $$X = C_0 C_1 \ldots C_p$$

Since each C_i is closed, X is closed and if $\xi \epsilon X$

then $$\xi \epsilon C_0, C_1, \ldots C_p$$

i.e., $$\xi \epsilon C, \xi/\lambda_1 \epsilon C, \ldots \xi/\lambda_p \epsilon C.$$

Hence X is the set of points ξ satisfying the required condition. So, we are only to show that the measure of X is positive.

Now,
$$X = C_0 C_1 \ldots C_p$$

$$\therefore \quad X' = \sum_{r=0}^{p} C_r' = \sum_{r=0}^{p} (U - C_r)$$

$$\therefore \quad X = U - \sum_{r=0}^{p} (U - C_r)$$

$$\therefore \quad |X| \geq |U| - \sum_{r=0}^{p} |U - C_r|$$

$$= |U| - \sum_{r=0}^{p} \{|U| - |C_r|\}.$$

Now,
$$|C_r| = \lambda_r^n |C| > |C|$$

$$\therefore \quad |U| - |C_r| < |U| - |C| \quad \text{for } r = 1, 2, \ldots p$$

$$\therefore \quad |X| > |U| - \sum_{r=0}^{p} \{|U| - |C|\}$$

$$= |U| - (p+1)(|U| - |C|)$$

$$> |U| - |C|$$

$$> 0.$$

This proves the Theorem.

If $p = 1$ and $n = 1$, then e is a linear closed set of positive measure in $-1 < x < 1$. So, by the above theorem there is a positive δ such that if α is any number satisfying $1 \leq \alpha < 1 + \delta$, then the set of points $\xi \epsilon C$ such that $\xi/\alpha \epsilon C$ is closed and is of positive measure. It follows therefore that there are infinity of pairs ξ and η belonging to C such that

$$\frac{|\xi|}{|\eta|} = \alpha.$$

Now, given a set E we consider the ratios $|\xi|/|\eta|$, where $\xi, \eta \epsilon E$ and $|\xi|, |\eta|$ are the distances of the points ξ, η from the origin Let us call the set of values of the above ratio for all possible pairs ξ, η of the points of E the ratio set R of E. It follows therefore from the above result that the ratio set R of a set of positive measure fills a continuum of which an end point is 1.

We have proved the above theorem by assuming that the set C is contained within the unit sphere.

Let us now suppose that the bounded closed set C of positive measure extends beyond the unit sphere.

Since C is bounded, there exists a positive b such that

$$|\xi| < b \quad \text{for} \quad \xi \varepsilon C.$$

Now, consider the set $C/b = D$. For any point $\xi' \varepsilon D$ we have $|\xi'| < 1$. So, the set D is closed, is of positive measure and is contained within the unit sphere.

So, given p there is a positive δ such that for any p positive numbers $\lambda_1, \lambda_2, \ldots \lambda_p$ satisfying $1 \leq \lambda_1 < \lambda_2 < \ldots < \lambda_p < 1+\delta$, the set of points $\xi' \varepsilon D$ such that

$$\frac{1}{\lambda_1}\xi', \frac{1}{\lambda_2}\xi', \ldots \frac{1}{\lambda_p}\xi'$$

belong to D is closed and is of positive measure.

Now, $\xi' \varepsilon D$ implies $b\xi' \varepsilon C$. So, the set of points $b\xi' \varepsilon C$ and such that

$$\frac{b\xi'}{\lambda_r} \varepsilon C \quad (r = 1, 2, \ldots p)$$

is closed and is of positive measure.

So, the result of Theorem 1 also remains valid if the set C extends beyond the unit sphere.

Theorem 2. *Let $\{\alpha_n\}$ be a sequence of real numbers such that $Lt\, \alpha_n = 1$, $\alpha_n > 1$, and let C be a closed bounded set of positive measure. Then there is a point $\xi \varepsilon C$ and a subsequence $\{\alpha_{n_r}\}$ of $\{\alpha_n\}$ such that $\{\xi/\alpha_{n_r}\}$ belong to C.*

Proof. There is a positive number δ_1 such that for any α_i, $1 < \alpha_i < 1+\delta_1 (p = 1$ in Theorem 1) the set $C_1 \subset C$ of points for which $\xi \varepsilon C$ and $\xi/\alpha_i \varepsilon C$ is closed and is of positive measure. We choose an α_i from the sequence $\{\alpha_n\}$ and call it α_{n_1}.

Proceeding with C_1 we have a positive number δ_2 such that for any α_j

$$1 < \alpha_j < 1+\delta_2$$

there is a set $C_2 \subset C_1$ where C_2 is closed and is of positive measure.

We choose α_j from the sequence $\{\alpha_n\}$ and call it α_{n_2} and it is possible to choose $\alpha_{n_2} < \alpha_{n_1}$.

Proceeding in this way we get a sequence of sets

$$C_1, C_2, C_3, \ldots$$

and a sequence of numbers

$$\alpha_{n_1}, \alpha_{n_2}, \ldots (\alpha_{n_r} \geq 1)$$

such that

$$C \supset C_1 \supset C_2 \supset \ldots$$

$$\alpha_{n_1} \geq \alpha_{n_2} > \alpha_{n_3} > \ldots$$

and

$$Lt\, \alpha_{n_r} = 1$$

Since each C_n is closed, bounded and of positive measure, so ΠC_r is non-empty.

So, there exists at least one ξ belonging to all C_n and for such a ξ

$$\frac{1}{\alpha_{n_r}}\xi \varepsilon C, \quad r = 1, 2, 3\ldots$$

This proves the theorem.

Theorem 3. *Let C be a closed bounded set of positive measure contained in the open unit sphere in E_n. Then it is possible to determine a positive null sequence $\{\mu_n\}$ such that if $\{\xi_n\}$ be any positive sequence satisfying $\xi_n < \mu_n (n = 1, 2, \ldots)$ then the set of points ξ of C such that*

$$\xi \,\varepsilon C \text{ and } \frac{\xi}{\lambda_n} \varepsilon C \text{ where } \lambda_n = 1 + \xi_n,$$

and $$n = 1, 2, \ldots$$

is closed and is of positive measure.

Proof. As shown in Kestelman's paper we can define a sequence $\{U_n\}$ of open sets satisfying

$$U_1 \supset U_2 \supset U_3 \supset \ldots, \; C = \prod_{n=1}^{\infty} U_n$$

and $$\sum_{n=1}^{\infty} | U_n - C | < | C |,$$

and such that all U_n are contained in the unit sphere.

If μ_n is the distance between C and the complement of U_n then $\{\mu_n\}$ is a positive null sequence.

Now, suppose that the sequence $\{\xi_n\}$ is such that $0 < \xi_n < \mu_n$ for every n. Then $\{\xi_n\}$ is also a null sequence and $\{\lambda_n\}$ is such that $\lambda_n = 1 + \xi_n > 1$, so $Lt\ \lambda_n = 1$.

Let $$C_r = C\lambda_r$$

Then $C_r \subset U_r$ for every r.

Let $$X = C_1 C_2 C_3 \ldots$$

If $\xi \varepsilon X$, then $\xi \varepsilon C_1, C_2, C_3, \ldots$

i.e., $$\frac{\xi}{\lambda_1} \varepsilon C, \; \frac{\xi}{\lambda_2} \varepsilon C, \; \frac{\xi}{\lambda_3} \varepsilon C, \ldots$$

The sequence $\{\xi/\lambda_r\}$ converges to ξ and since C is closed so $\xi \varepsilon C$.

Again each C_r is closed, so, X is closed. So X the set of points ξ satisfying the required condition.

So, we are only to show that the measure of X is positive.

Now,
$$X = C_1C_2C_3\ldots$$
$$= U_1 - \sum_{n=1}^{\infty}(U_1 - C_n)$$
$$= U_1 - \{\Sigma(U_1 - U_n) + \Sigma(U_n - C_n)\}$$
$$= \{U_1 - \Sigma(U_1 - U_n)\} - \Sigma(U_n - C_n)$$
$$= C - \Sigma(U_n - C_n)$$

Now,
$$|C_r| = \lambda_r^n |C| > |C| \quad \text{for all } r$$
$$\therefore \quad |U_n - C_n| < |U_n - C| \quad \text{for all } n$$
$$\therefore \quad |X| \geq |C| - \Sigma|U_n - C_n|$$
$$> |C| - \Sigma|U_n - C| > 0.$$

which proves the theorem.

Note. The results of the above theorems remain also valid if we replace the condition the set 'C is of positive measure' by 'C is a set of positive interior measure' since it is known that every set of positive interior measure contains a measurable subset of positive measure.

Theorem 4. *If E is a set which contains almost all points of an interval I, then E is universally permeable.*

Proof. Let Z be a subset of I of measure zero. Then it is sufficient to prove that $I - Z$ is universally permeable.

Let $\{\lambda_n\}$ be any sequence of real numbers, such that $Lt\ \lambda_n = 1$. We shall prove that the sequence permeates through the set $I - Z$, *i.e.* we shall prove that there is a $x \epsilon I$ such that all but a finite number of points of the sequence $\{x/\lambda_n\}$ belongs to $I - Z$.

Let $C_r = Z\lambda_r$ and $X = I \sum_{r=1}^{\infty} C_r$

Since $|Z| = 0$, so $|C_r| = 0$ for every r and so $|X| = 0$.

Suppose now that $x \epsilon I - X$

then
$$\frac{x}{\lambda_r} \text{not } \epsilon\, Z \quad \text{for any } r.$$

Since $x \epsilon I$, x is an interior point of I unless it is a boundary point. We conclude from this that $x/\lambda_r \epsilon I - Z$ for all sufficiently large values of r and this proves the theorem.

In the above theorem we have in fact proved that given any sequence $\{\lambda_n\}$ of real numbers satisfying $Lt\ \lambda_n = 1$ and a set E which contains almost all points of an interval I, then $x/\lambda_n \epsilon E$ for all large n and for almost all points x of I.

Now, considering the fact that if Z is a set of the first category, then $Z.\lambda_r$ where λ_r is a positive real number, is also a set of the first category, we may replace the set Z of the above theorem by the statement that 'Z is a set of the first category' and obtain the following theorem whose proof may be similarly given.

Theorem 5. *If E is a set which contains all points of an interval I except a set Z of the first category, then E is universally permeable.*

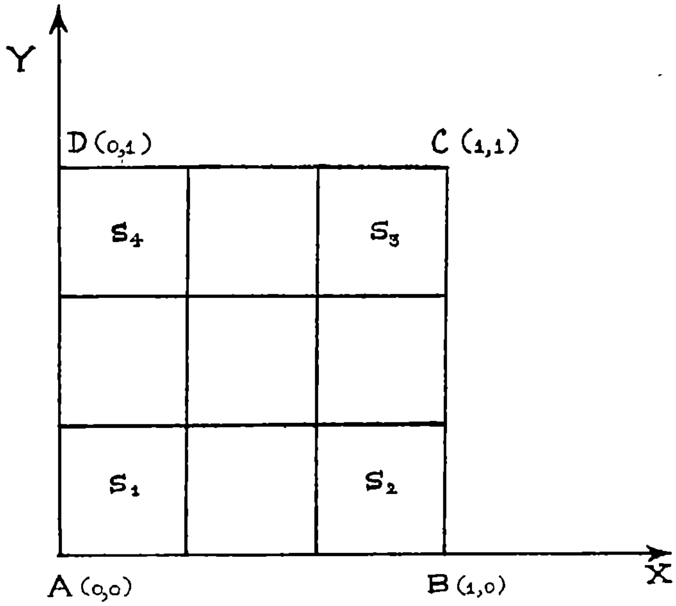
In the following, we shall consider the problem of finding a set of measure zero for which the ratio set R fills a continuum. It is known that [*cf.* J. F. Randolph] the distance set for the Cantor middle third set based on the unit interval fills the continuum $0 \leq x \leq 1$. The problem was first tackled by Steinhaus who used a geometrical method.

In a later paper, W. R. Utz has proved a more general result *i.e.* he has proved that if c and m are two real numbers such that $0 \leq c \leq 1$ and $\frac{1}{3} \leq |m| \leq 3$, then there exist numbers x and y of the Cantor middle third set on the unit interval [0, 1], such that $y - mx = c$.

We shall prove the following:

Theorem 6. *The ratio set R of Cantor's middle third set based on the unit interval* [0, 1] *fills a continuum.*

Proof. Let C_1 and C_2 denote the Cantor's middle third set based on the unit interval on the X and Y axis respectively of a rectangular cartesian coordinate system. The cartesian product of the sets C_1 and C_2 then evidently is contained in the closed square (0, 0), (1, 0), (1, 1), (0, 1). Let $ABCD$ denote the square and P denotes the product set. Then P consists of all points of the form (x, y) where $x \varepsilon C_1$, $y \varepsilon C_2$. To prove



the theorem we shall show that for suitable values of m, the straight line $y = mx$ passes through a point (x_1, y_1) of $P = C_1 \times C_2$ and $(x_1, y_1) \neq (0, 0)$.

For this, we divide AB into three equal segments by inserting lines parallel to the Y-axis. Similarly, we divide AD into three equal segments by drawing lines parallel to the X-axis.

We omit the portions of the straight lines thus drawn which lie outside the square $ABCD$. The square $ABCD$ then is divided into nine equal squares each of area the one-ninth of the original square. We delete the interior portion of the cross-shaped region thus formed and also the interior of the segments of the straight lines AB, BC CD, DA where the two interior strips of the square $ABCD$ parallel to the X and Y-axis respectively intersect these lines.

There remain only four equal closed squares, say S_1, S_2, S_3, S_4 at the four corners.

Now, if $1 \leq m \leq \frac{3}{2}$, then the straight line $y = mx$ intersects the square $ABCD$ and also the square S_3 which we rename as $A_1B_1C_1D_1$. We apply the same process on $A_1B_1C_1D_1$ and see that the straight line $y = mx$ with the above range of m will intersect one of the smaller closed square, say $A_2B_2C_2D_2 \subset A_1B_1C_1D_1$.

If the process is continued indefinitely, then we get a sequence of closed squares

$$A_1B_1C_1D_1 \supset A_2B_2C_2D_2 \supset \ldots \supset A_nB_nC_nD_n \supset \ldots$$

and the straight line $y = mx$ intersects each of them.

Since the sequence form a monotone decreasing sequence of closed squares, the product is non-empty. So, there exists a point (x_1, y_1) belonging to the product set and the straight line $y = mx$ passes through that point and since $(x_1, y_1) \epsilon A_1B_1C_1D_1$, $(x_1, y_1) \neq (0, 0)$.

It follows therefore that there exist points x and y of the Cantor middle third set based on the unit interval $(0, 1)$ such that $y/x = m$, $(y, x) \epsilon (0, 0)$, $1 \leq m \leq \frac{3}{2}$.

This proves the theorem.

In conclusion, I wish to express my sincere gratitude to Dr. H. M. Sengupta for his kind help and valuable guidance in the preparation of the paper.

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SOME NEW KERNELS FOR THE DERIVATION OF SELF-RECIPROCAL FUNCTIONS

By

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1. Introduction. Adopting the usual notation, we shall say that a function $f(x)$ is R_ν, if it is self-reciprocal for J_ν transforms, that is, if it satisfies the integral equation

$$f(x) = \int_0^\infty \sqrt{xy} \cdot J_\nu(xy) \cdot f(y) \cdot dy \qquad [R(\nu) > -1] \tag{1.1}$$

For $R_{\frac12}$ and $R_{-\frac12}$, we shall write R_s and R_c respectively.

A number of formulae for the derivation, from a known R_μ function, of another self-reciprocal function of a different order ν, say, has been given by several authors.

I will make use of the following result, given by Dr. Brij Mohan, (1932, § 8)

If $f(x)$ is R_μ, the function

$$g(x) = \int_0^\infty P(xy) \cdot f(y) \cdot dy$$

is R_λ, provided that

$$P(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^s \Gamma(\tfrac14 + \tfrac12\mu + \tfrac12 s)\Gamma(\tfrac14 + \tfrac12\lambda + \tfrac12 s)\chi(s) x^{-s} ds, \tag{1.2}$$

where $0 < c < 1$,

and
$$\chi(s) = \chi(1-s) \tag{1.3}$$

By the symmetry in μ and λ, in the above theorem, it can easily be deduced, that if $f(x)$ is R_λ, then $g(x)$ is R_μ. Here $P(x)$ is called a kernel transforming R_μ into R_λ, and vice-versa.

In the present paper, I have tried to investigate some new kernels by this rule, for the derivation of some R_ν functions.

2. Theorem 1: *Rule I. The kernel* $x^{\alpha-\nu+\frac12} e^{-x} L^{\alpha}_{\alpha-2\nu+1}(x)$ *transforms* $R_{\nu-1}$ *into* R_ν, *provided that* $\nu - \frac12 < \alpha < 0$, *where* $L^{\alpha}_{n}(x)$ *denotes a Laguerre polynomial of order* n.

To prove this, I start with the integral given by Howell (1937):

$$\int_0^\infty x^{l-1} e^{-x} L^{\alpha}_{n}(x) dx = \frac{\Gamma(l)\Gamma(n+1+\alpha-l)}{\Gamma(n+1)\Gamma(1+\alpha-l)},$$

provided that $l > 0$, $l > \alpha$, $n+1+\alpha-l > 0$, and n is a +ve integer.

Now, by Mellin's Inversion Formula (Hardy, 1918), we have

$$x^{\gamma}e^{-x}L_n^{\alpha}(x) = \frac{1}{2\pi i}\ \frac{2^{\gamma+n-1}}{\pi^{\frac12}\Gamma(n+1)}\int_{c-\gamma-i\infty}^{c-\gamma+i\infty} 2^{s}x^{-s}\delta\times$$

$$\frac{\Gamma(\frac12\gamma+\frac12 s)\Gamma(\frac12+\frac12\gamma+\frac12 s)\Gamma(\frac12+\frac12 n+\frac12\alpha-\frac12\gamma-\frac12 s)\Gamma(1+\frac12 n+\frac12\alpha-\frac12\gamma-\frac12 s)}{\Gamma(\frac12+\frac12\alpha-\frac12\gamma-\frac12 s)\Gamma(1+\frac12\alpha-\frac12\gamma-\frac12 s)}\,ds$$

Now, on putting $\gamma = \alpha-\nu+\frac12$, and $n = \alpha-2\nu+1$ we get

$$x^{\alpha-\nu+\frac12}e^{-x}L^{\alpha}_{\alpha-2\nu+1}(x) = \frac{1}{2\pi i}\int_{c-\frac12-\alpha+\nu-i\infty}^{c-\frac12-\alpha+\nu+i\infty} 2^{s}x^{-s}\Gamma(-\tfrac14+\tfrac12\nu+\tfrac12 s)\Gamma(\tfrac14+\tfrac12\nu+\tfrac12 s)\chi(s)\,ds,$$

where
$$\chi(s) = \frac{2^{2\alpha-3\nu+\frac12}\Gamma(\frac14+\frac12\alpha-\frac12\nu+\frac12 s)\Gamma(\frac34+\frac12\alpha-\frac12\nu-\frac12 s)\Gamma(\frac34+\frac12\alpha-\frac12\nu+\frac12 s)\Gamma(\frac14+\frac12\alpha-\frac12\nu-\frac12 s)}{\pi^{\frac12}\Gamma(n+1)\Gamma(-\frac14+\frac12\nu+\frac12 s)\Gamma(\frac14+\frac12\nu-\frac12 s)\Gamma(\frac14+\frac12\nu+\frac12 s)\Gamma(\frac34+\frac12\nu-\frac12 s)}$$

$$= \chi(1-s),$$

provided that $\nu-\frac12<\alpha<0$.

This integral is of the form (1.2), with $\mu = \nu$, and $\lambda = \nu-1$, in which $\chi(s)$ satisfied (1.3).

Hence the result.

Rule II. The kernel $x^{\alpha-\nu+\frac12}e^{-x}{}_1F_1(2\nu-1-\alpha;\alpha+1;x)$ *transforms* $R_{\nu-1}$ *into* R_{ν}, *provided that* $\nu-\frac12<\alpha<0$, *where* ${}_1F_1(\alpha;\beta;x)$ *denotes the generalised hypergeometric function.*

Using the formula

$$L_n^{\alpha}(x) = \frac{\Gamma(1+\alpha+n)}{n!\,\Gamma(\alpha+1)}{}_1F_1(-n;\alpha+1;x), \tag{α}$$

where n is a +ve integer, we get the result.

Rule III. The kernel $x^{\frac12\alpha-\nu}e^{-\frac12 x}W_{3\alpha/2-2\nu+3/2,\,\frac12\alpha}(x)$ *transforms* $R_{\nu-1}$ *into* R_{ν}, *provided that* $\nu-\frac12<\alpha<0$, *where* $W_{k,m}(x)$ *denotes the generalised Whittaker function.*

We arrive at the result by virtue of the relation

$$L_n^{m}(x) = \frac{(-1)^n}{n!}x^{-\frac12(m+1)}e^{\frac12 x}W_{n+\frac12 m+\frac12,\,\frac12 m}(x), \tag{β}$$

which is true for +ve integral values of n.

Rule IV. The kernel $x^{\alpha-\nu+\frac12}e^{-x}T_{\alpha}^{\alpha-2\nu+1}(x)$ *transforms* $R_{\nu-1}$ *into* R_{ν}, *provided that* $\nu-\frac12<\alpha<0$, *where* $T^{n}_{m}(x)$ *denotes a Sonine Polynomials of order* n.

We get the result by using the formula

$$T_m^n(x) = \frac{(-1)^n}{\Gamma(1+m+n)}L_n^m(x), \tag{δ}$$

which is true for +ve integral values of n.

Theorem II. *Rule V. The kernel* $x^{\nu-m-1}e^{\frac{1}{2}x}W_{3m-2\nu+\frac{1}{2},m}(x)$ *transforms* $R_{\nu-1}$ *into* R_ν, *provided that* $-\frac{1}{2}<m<\frac{1}{2}$, *and* $\nu>\frac{3}{2}(m+\frac{1}{2})$.

To prove this, we start with the integral given by Sinha (1942, § 1)

$$\int_0^\infty x^{l-1}e^{-\frac{1}{2}x}{}_2F_2(\alpha,\beta;\gamma,\delta;-xt)W_{k,m}(x)\,dx$$

$$=\frac{\Gamma(l+m+\frac{1}{2})\Gamma(l-m+\frac{1}{2})}{\Gamma(l-k+1)}\,{}_4F_3\left(\begin{matrix}\alpha,\beta,\ l+m+\frac{1}{2},\ l-m+\frac{1}{2}\\ \gamma,\ \delta,\ l-k+1\end{matrix};-t\right),$$

where $R(l\pm m+\frac{1}{2})>0$, and $|t|\leq 1$.

Taking $\alpha=\gamma$, $\beta=\delta$ and $t=-1$, we get the above integral in the form

$$\int_0^\infty x^{l-1}e^{\frac{1}{2}x}W_{k,m}(x)\,dx$$

$$=\frac{2^{l-k-2}\Gamma(\frac{1}{4}+\frac{1}{2}m+\frac{1}{2}l)\Gamma(\frac{3}{4}+\frac{1}{2}m+\frac{1}{2}l)\Gamma(\frac{1}{4}-\frac{1}{2}m+\frac{1}{2}l)\Gamma(\frac{3}{4}-\frac{1}{2}m+\frac{1}{2}l)\Gamma(-\frac{1}{2}k-\frac{1}{2}l)\Gamma(\frac{1}{2}-\frac{1}{2}k-\frac{1}{2}l)}{\pi^{3/2}\Gamma(\frac{1}{2}-k-m)\Gamma(\frac{1}{2}-k+m)}$$

where $R(l\pm m+\frac{1}{2})>0$ and $k+l$ is a negative integer.

Hence, by Mellin's Inversion Formula (3), we get

$$x^\gamma e^{\frac{1}{2}x}W_{k,m}(x)=\frac{2^{\gamma-k-2}}{\pi^{3/2}\Gamma(\frac{1}{2}-k-m)\Gamma(\frac{1}{2}-k+m)}\,\frac{1}{2\pi i}\times$$

$$\int_{c-\gamma-i\infty}^{c-\gamma+i\infty}2^s x^{-s}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}m+\tfrac{1}{2}\gamma+\tfrac{1}{2}s)\Gamma(\tfrac{3}{4}+\tfrac{1}{2}m+\tfrac{1}{2}\gamma+\tfrac{1}{2}s)\Gamma(\tfrac{1}{4}-\tfrac{1}{2}m+\tfrac{1}{2}\gamma+\tfrac{1}{2}s)\times$$

$$\Gamma(\tfrac{3}{4}-\tfrac{1}{2}m+\tfrac{1}{2}\gamma+\tfrac{1}{2}s)\Gamma(-\tfrac{1}{2}k-\tfrac{1}{2}\gamma-\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}-\tfrac{1}{2}k-\tfrac{1}{2}\gamma-\tfrac{1}{2}s)\,ds,\qquad (A).$$

where $-\frac{1}{2}-c<m<\frac{1}{2}+c$, and $k+l$ is a negative integer.

Now, putting $\gamma=\nu-m-1$ and $k=3m-2\nu+\frac{1}{2}$ we get

$$x^{\nu-m-1}e^{\frac{1}{2}x}W_{3m-2\nu+\frac{1}{2},m}(x)$$

$$=\frac{1}{2\pi i}\int_{c+m+1-\nu-i\infty}^{c+m+1-\nu+i\infty}2^s x^{-s}\Gamma(-\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\chi(s)\,ds,$$

where

$$\chi(s)=\frac{2^{3\nu-4m-7/2}\Gamma(-\frac{1}{4}-m+\frac{1}{2}\nu+\frac{1}{2}s)\Gamma(\frac{1}{4}-m+\frac{1}{2}\nu+\frac{1}{2}s)\Gamma(\frac{1}{4}-m+\frac{1}{2}\nu-\frac{1}{2}s)\Gamma(\frac{3}{4}-m+\frac{1}{2}\nu-\frac{1}{2}s)}{\pi^{3/2}\Gamma(2\nu-4m)\Gamma(2\nu-2m)}$$

$$=\chi(1-s),$$

provided that $-\frac{1}{2}<m<\frac{1}{2}$ and $\nu>\frac{3}{2}(m+\frac{1}{2})$.

This integral is of the form (1.2), with $\mu=\nu-1$ and $\lambda=\nu$, in which $\chi(s)$ satisfies (1.3).

Hence the result.

Using the formula

$$W_{k,\pm\frac{1}{4}}(x) = 2^{-k+\frac{1}{4}}x^{\frac{1}{4}}D_{2k-\frac{1}{2}}((2x)^{\frac{1}{2}}),$$

we get the following two rules:

Rule VII. The kernel $x^{\nu-1}e^{\frac{1}{2}x}D_{2-4\nu}((2x)^{\frac{1}{2}})$ transforms $R_{\nu-1}$ into R_ν, provided that $\nu>\frac{9}{8}$

Rule VIII. The kernel $x^{\nu-\frac{1}{2}}e^{\frac{1}{2}x}D_{-1-4\nu}((2x)^{\frac{1}{2}})$ transforms $R_{\nu-1}$ into R_ν, provided that $\nu\geq\frac{1}{2}$.

4. Theorem III. *Rule IX.* The kernel $x^{\nu+m-1}e^{\frac{1}{2}x}W_{-3m-2\nu+\frac{1}{2},\,m}(x)$ transforms $R_{\nu-1}$ into R_ν, provided that $\nu>\frac{3}{4}-\frac{3}{2}m$ and $-\frac{1}{2}<m<\frac{1}{2}$. To establish this rule, we put $\gamma=\nu+m-1$, and $k=\frac{1}{2}-2\nu-3m$, in the integral equation (A) of Rule VI and we get

$$x^{\nu+m-1}e^{\frac{1}{2}x}W_{\frac{1}{2}-2\nu-3m,\,m}(x)$$

$$=\frac{1}{2\pi i}\int_{c+1-\nu-m-i\infty}^{c+1-\nu-m+i\infty}2^s x^{-s}\Gamma(-\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\chi(s)ds,$$

where

$$\chi(s)=\frac{\Gamma(-\frac{1}{4}+m+\frac{1}{2}\nu+\frac{1}{2}s)\Gamma(\frac{1}{4}+m+\frac{1}{2}\nu-\frac{1}{2}s)\Gamma(\frac{1}{4}+m+\frac{1}{2}\nu+\frac{1}{2}s)\Gamma(\frac{3}{4}+m+\frac{1}{2}\nu-\frac{1}{2}s)}{2^{7/2-3\nu-4m}\pi^{3/2}\Gamma(2m+2\nu)\Gamma(4m+2\nu)}$$

$$=\chi(1-s),$$

provided that $\nu>\frac{3}{4}-\frac{3}{2}m$, and $-\frac{1}{2}<m<\frac{1}{2}$.

This integral is of the form (1 2) with $\mu=\nu-1$ and $\lambda=\nu$, in which $\chi(s)$ satisfies (1.3).

Hence the result.

Note: On putting $m=\pm\frac{1}{4}$, we get Rule VII and Rule VIII respectively.

5. Theorem IV. *Rule X.* The kernel $x^{-1}(1-e^{-x})$ transforms $R_{3/2}$ into R_3.

To prove this, we start with the following integral given by Brij Mohan (1942, § 2.1)

$$\int_0^\infty x^{p-1}K_m(ax)\mathcal{H}_n(bx)\,dx=\frac{2^{p-1}b^{n+1}\Gamma(\frac{1}{2}n+\frac{1}{2}m+\frac{1}{2}p+\frac{1}{2})\Gamma(\frac{1}{2}n+\frac{1}{2}p-\frac{1}{2}m+\frac{1}{2})}{\pi^{\frac{1}{2}}a^{n+p+1}\Gamma(n+\frac{3}{2})}\times$$

$${}_3F_2\left(\begin{matrix}1,\ \frac{1}{2}n+\frac{1}{2}m+\frac{1}{2}p+\frac{1}{2},\ \frac{1}{2}n-\frac{1}{2}m+\frac{1}{2}p+\frac{1}{2}\\ \frac{3}{2},\ n+\frac{3}{2}\end{matrix};-\frac{b^2}{a^2}\right)$$

where $n+p>|m|-1$ and $\mathcal{H}_n(bx)$ is Struve's function.

Putting $n=-\frac{1}{2}$ and $b=ia$, we get

$$\int_0^\infty x^{p-3/2}\sinh ax\,K_m(ax)dx=\frac{a^{\frac{1}{2}-p}\Gamma(\frac{1}{4}+\frac{1}{2}m+\frac{1}{2}p)\Gamma(\frac{1}{4}-\frac{1}{2}m+\frac{1}{2}p)\Gamma(\frac{1}{2}-\frac{1}{2}p)\Gamma(1-\frac{1}{2}p)}{2^{5/2}\Gamma(\frac{5}{4}-\frac{1}{2}m-\frac{1}{2}p)\Gamma(\frac{5}{4}+\frac{1}{2}m-\frac{1}{2}p)}$$

where $p>|m|-\frac{1}{2}$.

Hence, by Mellin's Inversion Formula (Hardy, 1918), we have

$$x^{\gamma-\frac{1}{2}} \sinh \tfrac{1}{2}x\, K_m(\tfrac{1}{2}x)$$

$$=\frac{2^{\gamma-3}}{2\pi i}\int_{c-\gamma-i\infty}^{c-\gamma+i\infty} 2^s x^{-s} \frac{\Gamma(\frac{1}{4}+\frac{1}{2}m+\frac{1}{2}\gamma+\frac{1}{2}s)\Gamma(\frac{1}{4}-\frac{1}{2}m+\frac{1}{2}\gamma+\frac{1}{2}s)\Gamma(\frac{1}{2}-\frac{1}{2}\gamma-\frac{1}{2}s)\Gamma(1-\frac{1}{2}\gamma-\frac{1}{2}s)}{\Gamma(\frac{5}{4}-\frac{1}{2}m-\frac{1}{2}\gamma-\frac{1}{2}s)\Gamma(\frac{5}{4}+\frac{1}{2}m-\frac{1}{2}\gamma-\frac{1}{2}s)}\,ds,$$

where $c > |m| - \frac{1}{2}$.

On putting $m = -\frac{1}{2}$ and $\gamma = 0$, we get

$$x^{-\frac{1}{2}} \sinh (\tfrac{1}{2}x) K_{-\frac{1}{2}}(\tfrac{1}{2}x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^s x^{-s} \Gamma(1+\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}s)X(s)\,ds,$$

where

$$X(s) = \frac{\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2}-\frac{1}{2}s)}{8\,\Gamma(1+\frac{1}{2}s)\Gamma(\frac{3}{2}-\frac{1}{2}s)} = \chi(1-s),$$

provided that $c > 0$.

This integral is of the form (1.2) with $\mu = \frac{3}{2}$ and $\lambda = \frac{1}{2}$, in which $X(s)$ satisfies (1.3). Hence the result.

I wish to express my respectful thanks to Dr. Brij Mohan, for his generous help and suggestions in the preparation of this note.

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ON THE DISTANCE SET OF THE CANTOR "MIDDLE THIRD" SET.

BY

N. C. BOSE MAJUMDAR, *Calcutta.*

(*Received—February, 18, 1958*)

Steinhaus (1920) first discussed the problem of distance sets. He shewed that for any linear set of positive interior measure, the distance set fills a continuum. The corresponding problem for sets in n-dimensional space was considered by Kestelman (1947) and he proved a number of elegant theorems about distance sets of sets in n-dimensional space and Steinhaus's theorem (1920) comes out as a particular case of one of his results. Sets for which the distance set fills a continuum have been called S-sets. The question has been asked if a set of measure zero may be an S-set. A simple example of such a set which is an S-set has been given by Kestelman (1947). He also gives another example of a set of zero measure which is not an S-set.

Steinhaus (1917) himself proved that the Cantor's (1883) set C is an example of a set of measure zero which is an S-set. In fact he proved that for every distance $d(0 \leq d \leq 1)$, there are pairs of points (x, y), $x \varepsilon C$ and $y \varepsilon C$, such that $y-x = d$. Cantor set C is therefore a special type of S-set where every distance between 0 and the diameter of the set viz., 1 is assumed. Such sets may be singled out from the class of S-sets and we prefer to call them SD-sets. The Cantor set C is therefore an SD-set.

The Cantor's set C was considered by Randolph (1940) also and he proved that it was an SD-set by using a method which was purely arithmetical. This was followed by another article by Randolph (1941) where he considered the necessary and sufficient conditions under which a nondense perfect set, defined in a particular manner could be an SD-set.

The question of the Cantor set C was subsequently taken up by Utz (1951) and he solved a somewhat more general problem than that of Steinhaus (1917) and Randolph (1940, '41).

The following question may now be considered. Consider the Cantor's set C. It is known that it is an SD-set, *i.e.*, given any $d(0 \leq d \leq 1)$, there exists a pair of points x and y such that $x \varepsilon C$ and $y \varepsilon C$ and such that $y-x = d$. We now propose to enquire about the *power* of the aggregate of such pairs (x, y) for any particular d. It should be noted that Randolph (1940) conjectured that the same distance d may be attained for more than one pair of points and in particular he constructed an example to show that a distance was assumed for more than one pair of points.

In the present article the author considers the problem of the power of the aggregate of pairs of point x and y, $x \epsilon C$, $y \epsilon C$ such that $y-x = d$, where $0 \leq d \leq 1$ and his main result is embodied in the two following theorems :

Theorem I. *Except possibly for an enumerable set of values of* d $(d \epsilon C)$, *the aggregate of pairs of points* x *and* y, $x \epsilon C$, $y \epsilon C$ *and such that* $y-x = d$, *has the power of the continuum.*

Theorem II. *For each of an everywhere dense set of values of* d $(d \epsilon cC$, *i.e ,* d *belonging to the set, complementary to the Cantor set* C), *the aggregate of pairs* (x, y), $x \epsilon C$, $y \epsilon C$, *such that* $y-x = d$, *has the power of the continuum.*

I.

We first consider the Theorem I.

The proof of the theorem will be given in a number of steps, (*a*), (*b*) and (*c*) which follow :

Following a suggestion given to Randolph (1940) by Hurwitz, the representation of a number in $0 \leq d \leq 1$, in the scale of 3 is broken into a number of blocks in the following fashion. The representation of any number will in general consist of the digits 0, 1 and 2. Now starting from left, we take the first block of 0's and 2's and call it δ_1 ; it may, however, so happen, that this block is absent, then d begins with 1. It may happen that δ_1 is not followed by any digit, then $d = \cdot\delta_1$ Now, if δ_1 is followed by 1, two cases may arise ; (*i*) 1 is not followed by any more digits, then $d = \cdot\delta_1 1$, or (*ii*) 1 is followed by a block of 0's and 2's, which we call δ_2 (which may however be absent, in which case 1 is followed by another 1), and so on. A whole block of δ may however be unending.

(*a*) We now take up a number $d (0 \leq d \leq 1)$ such that $d = \cdot\delta_1$. If δ_1 consists of finite number of 0's and 2's, then it represents the length of a segment whose right hand end point is the right hand end point of a contiguous interval of C, if the left hand end point of the segment coincides with the point 0.

Denoting $d = \cdot\delta_1$, we write $x = \cdot(0)_1$ and $y = \cdot\delta_1$, where $(0)_1$ is obtained from δ_1 by retaining its 0's and replacing its 2's by 0's. Thus $x \epsilon C$ and $y \epsilon C$ such that $y-x = \cdot\delta_1$.

Now if we put an unending complex of 0's and 2's to the right of $(0)_1$, we get an $x' \epsilon C$ and on putting the same complex to the right of δ_1 (in y), we get an $y' \epsilon C$, and such that $y'-x' = d$ again. But as the aggregate of such complexes of 0's and 2's has the power of the continuum, the number of pairs of points x and y such that $x \epsilon C$ $y \epsilon C$ and $y-x = d$, has the power of the continuum.

(*b*) Take a $d = \cdot\delta_1 1$.

By the previous device it would coincide with the lefthand end point of a complementary interval of C.

Take $x = \cdot(0)_1 1$ and $y = \cdot\delta_1 2$, then $x \epsilon C$, $y \epsilon C$ and $y - x = d$. Now if we suffix an infinite complex of 0's and 2's to the right of x and y, the operation throws one of them viz., x out of C. Hence this process does not provide us with an infinity of pairs of (x, y) such that $x \epsilon C$, $y \epsilon C$ and $y - x = d (= \cdot\delta_1 1)$.

Now we can give a lower bound of the number of such pairs of (x, y), $x \epsilon C$, $y \epsilon C$ and $y - x = d$, as follows: Let the number of 0's in δ_1 be denoted by the symbol ${}^{\circ}\delta_1$. Let us now choose an $x \epsilon C (x + d$ not greater than 1), in such a manner that $x + d (= y) \epsilon C$. Number of such choices of x is at least $= 3.2^{\circ\delta_1}$. [Proof: Let $d = \cdot\delta_1 1 = \cdot c_1 c_2 \ldots c_{n-1} 1$, where $c_k = 0$ or 2. When c_k is 2, choose in its place, in making an $x (\epsilon C)$, 0 and when $c_k = 0$, choose in its place, in the construction of x either 0 or 2; in place of $c_n (= 1)$ choose (in making an x) 0 or 1 or 2. Therefore, it follows, that the number of pairs of points (x, y), $x \epsilon C$, $y \epsilon C$ and $y - x = d = \cdot\delta_1 1$ is at least $3\ 2^{\circ\delta_1}$. (*e.g.*, when $d = \frac{1}{3^n} = \cdot 000 \ldots 01$, the number of pairs of points $= 3.2^{n-1}$; when $d = \frac{2}{3} + \frac{1}{3^3} = \cdot 201$, the number of pairs of points $= 3.2^1 = 6$)].

But the aggregate of d of this type $(d = \cdot\delta_1 1)$ is enumerable.

(*c*) If d is a number such that a segment of length d with an end point $x = 0$ coincides with an external point of C, then $d = \cdot\delta_1$, where δ_1 is a single unending block of 0's and 2's, such that each of 0 and 2 occurs infinitely often in δ_1.

Let $d = \cdot\delta_1 = \cdot 0200220002002 \ldots\ldots$, say.

To get x, we may put 0 below each 2 (in d) and at random put either a 0 or 2 below each 0. Adding such an x with d we get a y, such that in the representation of y there is a 2 corresponding to each 2 of d and either 0 or a 2 inplace of a 0 in d. As such, $x \epsilon C$, $y \epsilon C$ and $y - x = d$. So, we have, that the aggregate of pairs (x, y), such that $x \epsilon C$, $y \epsilon C$ and $y - x = d$ of the above type, has the power of the continuum.

This completes the proof of the statement made in Theorem I.

II.

We now propose to prove Theorem II, given in a number of steps (*A*), (*B*) and (*C*) which follow:

Let $d \epsilon cC$ (*i.e.*, we take a d belonging to the set complementary to C). The representation of such a d will contain at least one 1, provided that, if it contains a single 1, then it cannot be the last significant digit in the representation. So, if d ends with a 1, then it must contain at least one more 1; *e.g.*, $d = \cdot 11$; $d = \cdot 010202$.

(*A*) Let d contain an even number of 1's. Then d is of the form, $d = \cdot\delta_1 1 \delta_2 1 \delta_3 1 \delta_4 1 \ldots \delta_{2n-1} 1 \delta_{2n} 1 \delta_{2n+1}$, where one or more of δ's may be void. Three cases may appear under this head:

(*i*) δ_{2n+1} is void; then d ends with a 1.

(*ii*) δ_{2n+1} is finite; then d ends with a 2.

(*iii*) δ_{2n+1} is an infinite complex of 0's and 2's.

(*i*) Let δ_{2n+1} be void.

Writing $d = \cdot\delta_1 1\delta_2 1\ \delta_3 1\delta_4 1\ \delta_5 1\delta_6 1\ldots\delta_{2n-1}1\ \delta_{2n}1$, we may write

$x = \cdot(0)_1 0\delta'_2 2\ (0)_3 0\delta'_4 2\ (0)_5 0\delta'_6 2\ldots(0)_{2n-1}0\delta'_{2n}2$, where $(0)_k$ has been obtained from δ_k by retaining all its zeros and at the same time its 2's have been replaced by 0's, and δ'_k has been obtained from δ_k by replacing in it each 0 by a 2 and each 2 by a 0.

Then $y = x+d = \cdot\delta_1 2(0)_2 0\ \ \delta_3 2(0)_4 0\ \ \delta_5 2(0)_6 0\ldots\ldots\delta_{2n-1}2(0)_{2n}0$

So the pair (x, y) is such that $x\epsilon C$, $y\epsilon C$ and $y-x = d$. Now we may put any complex of infinity of 0's and 2's to the right of above x and y and get x' and y', such that $x'\epsilon C$, $y'\epsilon C$ and $y'-x'(\equiv y-x) = d$. As the aggregate of infinite complexes of 0's and 2's has the power of the continuum, we conclude that the aggregate of (x, y) such that $x\epsilon C$, $y\epsilon C$ $y-x = d$, where d is of the above form has the power of the continuum.

(*ii*) Let δ_{2n+1} be non-empty and finite.

Let $d = \cdot\delta_1 1\delta_2 1\ \delta_3 1\delta_4 1\ \delta_5 1\delta_6 1\ldots\delta_{2n-1}1\delta_{2n}1\ \delta_{2n+1}$, and we write

$$x = \cdot(0)_1 0\delta'_2 2\ \ (0)_3 0\delta'_4 2\ldots(0)_{2n-1}0\delta'_{2n}2\ \ (0)_{2n+1}.$$

Then $\qquad y = x+d = \cdot\delta_1 2(0)_2 0\ \ \delta_3 2(0)_4 2\ldots\delta_{2n-1}2\ \ (0)_{2n}0\ \ \delta_{2n+1}.$

We have thus a pair (x, y), such that $x\epsilon C$, $y\epsilon C$ and $y-x = d$. As the previous remark made at the end of the case (*i*) applies here also, we conclude that the aggregate of pairs (x, y) such that $x\epsilon C$, $y\epsilon C$ and $y-x = d$, where d is of the form mentioned in (*ii*) has the power of the continuum.

(*iii*) Let δ_{2n+1} be an infinite complex of 0's and 2's.

Let $\qquad d = \cdot\delta_1 1\delta_2 1\ \ \delta_3 1\delta_4 1\ \ \delta_5 1\delta_6 1\ldots\ldots\delta_{2n-1}1\delta_{2n}1\ \delta_{2n+1}.$

We take $\qquad x = \cdot(0)_1 0\delta'_2 2\ \ (0)_3 0\delta'_4 2\ \ (0)_5 0\delta'_6 2..\ (0)_{2n-1}0\delta'_{2n}2\ \ \delta''_{2n+1}$,

where δ''_{2n+1} is obtained from δ_{2n+1} by changing every 2 in it, into a 0 and by either keeping a 0 in it as it is or changing it into 2 at will. It follows that the aggregate of complexes $\{\delta''_{2n+1}\}$ has the power of the continuum. Fixing one δ''_{2n+1} we write,

$$y = x+d = \cdot\delta_1 2(0)_2 0\ \ \delta_3 2(0)_4 0\ldots\delta_{2n-1}2(0)_{2n}0\ \delta'''_{2n+1},$$

where $\delta'''_{2n+1}(\equiv\delta_{2n+1}+\delta''_{2n+1})$ is obtained from the complex δ_{2n+1} by keeping every 2 in it as it is and changing a 0 in it either into 2 or keeping it unchanged; so both the complexes δ''_{2n+1}, δ'''_{2n+1} are made of 0's and 2's only. For each choice of δ''_{2n+1} we thus get a pair (x, y) such that $x\epsilon C$, $y\epsilon C$ and $y-x = d$. As the aggregate of complexes $\{\delta''_{2n+1}\}$ has the power of the continuum, the aggregate of such pairs of (x, y) has also the power of the continuum.

(*B*) Let d contain an odd number of 1's. Then d is of the form, $d = \cdot\delta_1 1\delta_2 1$ $\delta_3 1\delta_4 1\ \ \delta_5 1\delta_6 1\ldots\delta_{2n-3}1\delta_{2n-2}1\ \ \delta_{2n-1}1\ \ \delta_{2n}$, where one or more of δ's may be void. We shall discuss three cases according as, (*i*) δ_{2n} is void,

(*ii*) δ_{2n} is non-empty and finite,

and (*iii*) δ_{2n} is an infinite complex of 0's and 2's.

(*i*) Let δ_{2n} be void and let

$$d = \cdot\delta_1 1\delta_2 1 \quad \delta_3 1\delta_4 1 \quad \delta_5 1\delta_6 1 \ldots \delta_{2n-3} 1 \quad \delta_{2n-2} 1 \quad \delta_{2n-1} 1\,;$$

we may take $x = \cdot(0)_1 0\delta'_2 2 \quad (0)_3 0\delta'_4 2 \quad (0)_5 0\delta'_6 2 \ldots (0)_{2n-3} 0 \quad \delta'_{2n-2} 2 \quad (0)_{2n-1} 0,$

and then $y = x+d = \cdot\delta_1 2(0)_2 0 \quad \delta_3 2(0)_4 0 \quad \delta_5 2(0)_6 0 \ldots \delta_{2n-3} 2(0)_{2n-2} 0 \quad \delta_{2n-1} 1.$

The process of putting an infinite complex of 0's and 2's at the ends of x and y will throw y out of C. So the above process gives only one pair of (x, y) such that $x \epsilon C$, $y \epsilon C$ and $y-x=d$.

We may however, change any 0 if it exists in a δ_{2k-1} into either 0 or 2 while changing a 2 in δ_{2k-1} into a 0 and changing any 2 if it exists in δ_{2k} into either 0 or 2 while chagning 0 in it (*i.e.* δ_{2k}) if it exists, into a 2 without throwing any of x and y out of C, while keeping $y-x = d$. And the number of ways in which this may be done is $2^{\Sigma({}^0\delta_{2k-1}+{}^2\delta_{2k})}$, where ${}^0\delta_{2k-1}$ = the number of 0's in δ_{2k-1} and ${}^2\delta_{2k}$ = the number of 2's in δ_{2k}. So there is at least $2^{\Sigma({}^0\delta_{2k-1}+{}^2\delta_{2k})}$ pairs of (x, y), where $x \epsilon C$, $y \epsilon C$ and such that $y-x = d$.

We may also write

$$d = \delta_1 1\delta_2 1 \ \delta_3 1\delta_4 1 \ldots \delta_{2n-3} 1\delta_{2n-2} 1 \ \delta_{2n-1} 1,$$

$$x = \cdot(0)_1 0\delta'_2 2 \ (0)_3 0\delta'_4 2 \ldots (0)_{2n-3} 0\delta'_{2n-2} 2 \ (0)_{2n-1} 1,$$

and therefore, $y = x+d = \cdot\delta_1 2(0)_2 0 \ \delta_3 2(0)_4 0 \ \delta_5 2(0)_6 0 \ldots \delta_{2n-3} 2(0)_{2n-2} 0\delta_{2n-1} 2.$

The process of putting a complex of 0's and 2's at the ends of x and y throws one of them (namely x) out of C. And as the process of changing 0's in δ_{2k-1} into 2's and 0's and changing its 2's into 0's and changing 0's in δ_{2k} into 2's and changing its 2's either into 0's or 2's, apply here also as above, keeping x and y in C and making $y-x = d$, we conclude that the above process gives again at least $2^{\Sigma({}^0\delta_{2k-1}+{}^2\delta_{2k})}$ new pairs of (x, y), such that $x \epsilon C$ and $y \epsilon C$ and $y-x = d$. Again, if we take,

$$d = \cdot\delta_1 1\delta_2 1 \quad \delta_3 1\delta_4 1 \ldots \delta_{2n-3} 1 \quad \delta_{2n-2} 1 \quad \delta_{2n-1} 1,$$

and $x = \cdot(0)_1 0\delta'_2 2 \quad (0)_3 0\delta'_4 2 \ldots\ldots (0)_{2n-3} 0\delta'_{2n-2} 2(0)_{2n-1} 2,$ then we observe

that if, δ_{2n-1} be void, 1 appears in y at the place corresponding to 1 preceding the last 1 in d. If δ_{2n-1} consists of only 2's even then 1 appears in y in the same place. If however δ_{2n-1} contains only 0's, in the representation of y, 1 appears in the place of 0 in δ_{2n-1} which is closest to the last 1 in d and then 1 doesnot occur in any other place in the representation of y besides this 1. If δ_{2n-1} contains 0's and 2's both, then the corresponding $y (= x+d)$ will have 1 at the place of 0 in δ_{2n-1} closest to the last 1 in d, followed by all 0's. In any of the above cases, representation of y, will therefore end with a 1.

Following the same argument as is given in the above two cases, we can determine at least $2^{\Sigma({}^0\delta_{2k-1}+{}^2\delta_{2k})}$ new pairs of (x, y) other than the two previous sets, where $x\epsilon C$, $y\epsilon C$ and $y-x = d$. Therfore the total number of pairs of (x, y) under this head is at least $3.2^{\Sigma({}^0\delta_{2k-1}+{}^2\delta_{2k})}$.

(*ii*) Let δ_{2n} be non-empty and finite.

Let $$d = \cdot\delta_1 1\delta_2 1 \quad \delta_3 1\delta_4 1\ldots\delta_{2n-3}1\delta_{2n-2}1 \quad \delta_{2n-1}1\delta_{2n}.$$

As in the above cases in each $\delta_{2k-1}(k = 1, 2,\ldots, n)$ we replace 0's by (0's or 2's) and 2's by 0's, and also in each $\delta_{2k}(k = 1, 2. 3\ldots, n-1)$ we replace 0's by 2's and 2's by (0's or 2's). We get some x's (each ϵC and each giving rise to a corresponding $y\epsilon C$, such that $y-x = d$), the number of which $= n_1 = 2^{\sum_{k=1}^{n} {}^0\delta_{2k-1}+\sum_{k=1}^{n-1} {}^2\delta_{2k}}$, provided we retain 1 in the last incomplete group $\delta_{2n-1}1\,\delta_{2n}$ and replace δ_{2n} by $(0)_{2n}$. Now in each such x, we replace 1 by 0 in the last incomplete group and revive δ_{2n} [*i.e* replace $(0)_{2n}$ by δ_{2n}], wherein we replace 2's by (0's or 2's) and 0's by 2's,—but it should be noted that the last 2 in δ_{2n} must be retained (and must not be replaced by 0) In this way, we have so far determined x's (each ϵC and each giving rise to a corresponding $y\epsilon C$ such that $y-x = d$); the number of which $= n_1 + n_1.2^{{}^2\delta_{2n}-1} = n_1 + n_1 n_2$, where $n_2 = 2^{{}^2\delta_{2n}-1}$

Now consider any one x belonging to the set n_1 in each of which δ_{2n} is revived. In each such x replace 1 preceding δ_{2n} by 0. And then, in δ_{2n}, replace the first 2 by 1 which should be followed by all zeros and all zeros preceding this 1 should each be replaced by 2. We get some new x's satisfying the condition of the problem. Replace the second 2 in δ_{2n} by 1 followed by all zeros, whereas each zero preceding this 1 in δ_{2n} should be replaced by 2 and each 2 by 2 or 0. We again get some new x's. We continue like this until we reach the last 2 in δ_{2n} (*i e.* ${}^2\delta_{2n}$ th 2), which we replace by 1, where as each zero preceding this 1 in δ_{2n} is replaced by 2 and each 2 by 2 or 0. We get some new x's.

Hence the number of new x's produced by one old x, belonging to the set

$$n_1 = 1+2+2^2+\ldots+2^{{}^2\delta_{2n}-1}$$
$$= 2^{{}^2\delta_{2n}}-1 = 2n_2-1.$$

Hence in all the total number of x's (each ϵC and each giving rise to $y\epsilon C$ and $y-x = d$) under this head $= N = n_1 + n_1 n_2 + n_1(2n_2-1)$

$$= 3n_1 n_2 = 3.2^{\sum_{k=1}^{n}({}^0\delta_{2k-1}+{}^2\delta_{2k})-1}.$$

We have thus, at least N pairs of points x and y, $x\epsilon C$, $y\epsilon C$ such that $y-x=d$, where d is of the type mentioned in (*ii*).

Illustration: Let $d = \cdot 20 \quad 1 \quad 02022 \equiv \delta_1 1\delta_2$

$\therefore N = 3.2^{1+3-1} = 24.$

$d = \cdot 20 \quad 1 \quad 02022$

$$\therefore\ x = \left.\begin{array}{lll} \cdot 00 & 1 & 00000 \\ \cdot 02 & 1 & 00000 \end{array}\right\} \quad \begin{array}{l} \therefore\ y = \cdot 20\quad 2\quad 02022 \\ n_1 = 2 \end{array}$$

$$\left.\begin{array}{lll} \cdot 00 & 0 & 20202 \\ \cdot 00 & 0 & 20222 \\ \cdot 00 & 0 & 22202 \\ \cdot 00 & 0 & 22222 \\ \cdot 02 & 0 & 20202 \\ \cdot 02 & 0 & 20222 \\ \cdot 02 & 0 & 22202 \\ \cdot 02 & 0 & 22222 \end{array}\right\} \quad \begin{array}{l} \therefore\ y = \cdot 20\quad 2\quad 00001 \\ \\ n_1 n_2 = 2 \times 4 = 8 \end{array}$$

Consider now the first of n_1, viz., $\cdot 00\quad 1\quad 00000$.

We revive δ_{2n}, and thus we get $\cdot 00\quad 1\quad 02022$.

Following the instructions, given above, we get some new x's, given by,

$$x = \left.\begin{array}{lll} \cdot 00 & 0 & 21000 \\ \cdot 00 & 0 & 22210 \\ \cdot 00 & 0 & 20210 \\ \cdot 00 & 0 & 22221 \\ \cdot 00 & 0 & 22201 \\ \cdot 00 & 0 & 20221 \\ \cdot 00 & 0 & 20201 \end{array}\right\} \quad \begin{array}{l} y = \cdot 20\quad 2\quad 00022 \\ \\ \therefore\ N = 2 + 8 + 2 \times 7 = 24. \end{array}$$

so on, there being $2 (= n_1)$ such groups each of $2^{2\delta_1 - 1} = 2^3 - 1 = 7x$'s.

(*iii*) Let δ_{2n} be an infinite complex of 0's and 2's and

let $$\delta_{2n} = 22020222022002200022.\ \textit{ad inf}, \text{ say}.$$

Consider $$d = \cdot\delta_1 1 \delta_2 1\quad \delta_3 1 \delta_4 1 \ldots \delta_{2n-3} 1 \delta_{2n-2} 1\quad \delta_{2n-1} 1\quad \delta_{2n}$$

and $$x(\epsilon C) = \cdot(0)_1 0 \delta'_2 2\quad (0)_3 0 \delta'_4 2\ ..(0)_{2n-3} 0\quad \delta'_{2n-2} 2\quad (0)_{2n-1} 0\quad \delta''_{2n},$$

where δ''_{2n} is obtained from δ_{2n} by replacing its 0's by 2's and changing any number of its 2's by 2's or 0's at will. Keeping a δ'_{2n} fixed, we have

$$y = x + d = \cdot\delta_1 2(0)_2 0\ \ \delta_3 2(0)_4 0 \ldots \delta_{2n-3} 2\quad (0)_{2n-2} 0\ \ \delta_{2n-1} 2\ \ \delta'''_{2n},$$

where $\delta'''_{2n} (= \delta_{2n} + \delta''_{2n})$ is obtained from δ_{2n} by keeping its every 0 and replacing any number of its 2's either by a 0 or by a 2 at will. Now the set $\{\delta''_{2n}\}$ has the power of the continuum. Therefore the aggregate of the pairs of points (x, y), $x \epsilon C$ and $y \epsilon C$ such that $y - x = d$ (of the above form) has the power of the continuum.

(*C*) Let d contain an infinity of 1's.

Then d is of the form, $d = \cdot\delta_1 1 \delta_2 1\quad \delta_3 1 \delta_4 1\quad \delta_5 1 \delta_6 1 \ldots \ldots.,$ containing the groups $(\delta 1 \delta 1)$ in unending succession.

We have, $$d = \cdot\delta_1 1 \delta_2 1\quad \delta_3 1 \delta_4 1\quad \delta_5 1 \delta_6 1 \ldots\ldots,$$

$$x = \cdot(0)_1 0 \delta'_2 2\quad (0)_3 0 \delta'_4 2\quad (0)_5 0 \delta'_6 2\ \ldots..,$$

and therefore $$y = x+d = \cdot\delta_1 2(0)_2 0\ \delta_3 2(0)_4 0\ \ \delta_5 2(0)_6 0\ldots\ldots$$

We thus have a pair (x, y), such that $x \epsilon C$, $y \epsilon C$ and $y-x = d$.

(*i*) If an infinity of δ_{2n-1}'s contain 0's or an infinity of δ_{2n}'s contain 2's (or if both), we can in δ_{2n-1} keep 2's unchanged and change 0 into 2 or keep it unaltered at will (or we can in δ_{2n} keep 0 unchanged and change 2 into 0 or keep it unaltered at will), without making x or $y(=x+d)$ go out of C and thus $y-x = d$; as we can perform the above operation in a number of ways having the power of the continuum, we have the following theorem:

If a d is given by, $d = \cdot\delta_1 1\delta_2 1\ \delta_3 1\delta_4 1\ldots\delta_{2n-1}1\delta_{2n}1\ \ldots$, where an infinity of δ_{2n-1}'s have the digit 0 or an infinity of δ_{2n}'s have the digit 2, then the aggregate of pairs (x, y), such that $x \epsilon C$, $y \epsilon C$ and $y-x = d$, has the power of the continuum.

N B. As the set of d's for which an infinity of δ_{2n-1}'s contain at least one 0 (in each δ_{2n-1}) and (or) an infinity of δ_{2n} contain at least one 2 (in each δ_{2n}), is dense in $0 \leq x \leq 1$, the above result [even if we disregard other d's ϵC] shows the truth of Theorem II.

(*ii*) Suppose now, that, $d = \cdot\delta_1 1\delta_2 1\ \delta_3 1\delta_4 1\ldots\ldots$, is such that at best a finite number of δ_{2n-1} have 0's, and at best a finite number of δ_{2n}'s have 2's, the above process gives precisely $2^{\Sigma({}^0\delta_{2n-1}+{}^2\delta_{2n})}$ number of pairs of (x, y), $x \epsilon C$, $y \epsilon C$ and $y-x = d$. We may say that we can find at least $2^{\Sigma({}^0\delta_{2n-1}+{}^2\delta_{2n})}$ number of pairs of (x, y) of the above description.

(*iii*) If no δ_{2n-1} contains a 0 and no δ_{2n} contains a 2, then the above process gives $2^{0+0} = 1$ pair of (x, y) such that $x \epsilon C$, $y \epsilon C$ and $y-x = d$. We believe that this is the exact number of pairs of (x, y) fulfilling the above conditions.

We take the particular case, where $d = \cdot 1111\ldots\ldots$ad inf. This is a case covered by the above description (*iii*) of d and the above process gives only one pair of (x, y), $x \epsilon C$, $y \epsilon C$ and $y-x = d$,

viz., $$d = \cdot 111111\ldots\ldots \text{ad inf.}, \ (=\tfrac{1}{2}),$$

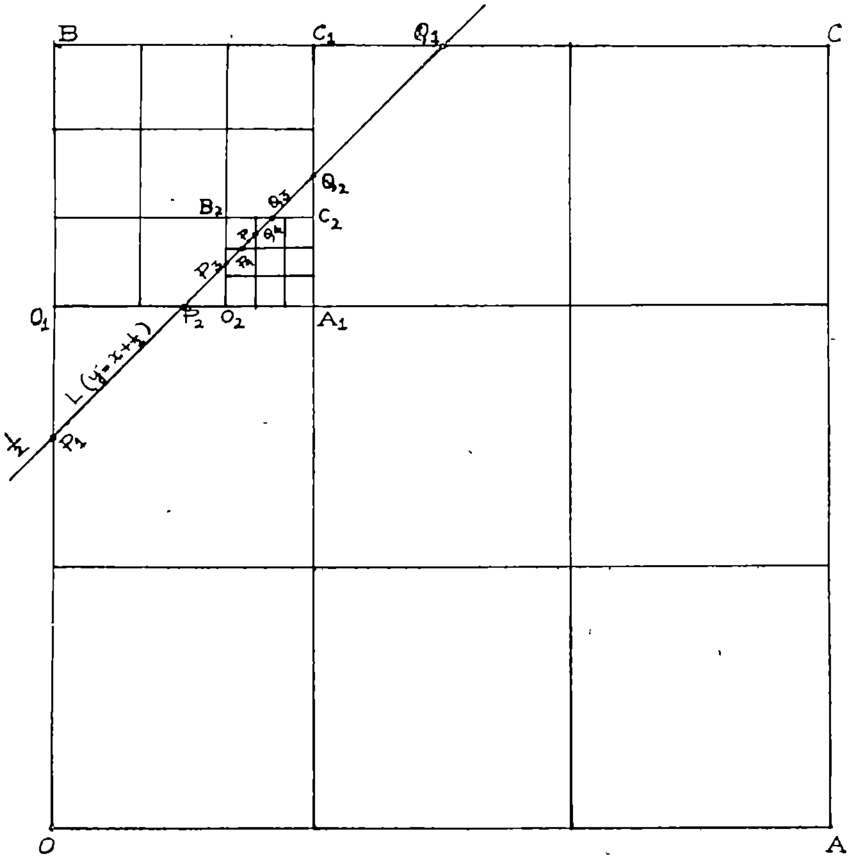
$$x = \cdot 020202\ldots\ldots \text{ad inf.}, \ (=\tfrac{1}{4}),$$

and $$\therefore\ y = x+d = \cdot 202020\ldots\ldots \text{ad inf.}, \ (=\tfrac{3}{4}).$$

We propose to show that in this particular case there is one and only one pair of (x, y) as calculated above, such that $x \epsilon C$, $y \epsilon C$ and $y-x = d$.

Let us consider the straight line $L(y = x+\frac{1}{2})$. It cuts the unit square $S(OACB)$ [(0, 0); (1, 0); (1, 1); (0, 1)] at P_1 and Q_1 where $P_1Q_1 = 1/2^{\frac{1}{2}}$ and P_1, Q_1 are the points of bisection of the left hand and upper sides of the square S. We shall denote the Cantor's set on OA by C_x and that on OB by C_y and their product set by $E(\equiv C_x \times C_y)$. There is no point of the set E contained in the unit square S, outside P_1Q_1 which are common to L $(y = x+\frac{1}{2})$ and the unit square S.

At the first stage (of suppression of intervals on the x and y axes) of formation of E by constructing C_x and C_y on the x and y axes respectively, we are left with 4 squares in the original unit square S; the line L cuts the NW corner square $S_1(O_1A_1C_1B)$ at P_2, Q_2, which are the points of bisection of the bottom and the right hand sides of this square S_1. Coordinates of P_2 ar $(\frac{1}{6}, \frac{2}{3})$ and those of Q_2 are $(\frac{1}{3}, \frac{5}{6})$. $\therefore$ $P_2Q_3 = 1/3(2)^{\frac{1}{2}}$.



Obviously, there is no point of E outside P_2Q_2 which are common to L and the original unit square S.

At the second stage of formation of the set $E(= C_x \times C_y)$ we are left with 4 squares (in the single square S_1) and the line L cuts the SE corner square $S_2(O_2A_1C_2B_2)$ at P_3 and Q_3, which are the points of bisection of the lefthand and upper sides of the square S_2. Coordinates of P_3 are $(\frac{2}{9}, \frac{13}{18})$ and those of Q_3 are $(\frac{5}{18}, \frac{7}{9})$ and therefore $P_3Q_3 = 1/3^2 2^{\frac{1}{2}}$.

Obviously, there is no point of E outside P_3Q_3 which are common to L and the original unit square S. This process is continued indefinitely and thus we have a succession of closed segments $P_1Q_1 \supset P_2Q_2 \supset P_3Q_3 \supset P_4Q_4 \ldots$, of respective lengths $1/2^{\frac{1}{2}}$, $1/3 2^{\frac{1}{2}}$, $1/3^2 2^{\frac{1}{2}}$, $1/3^3 2^{\frac{1}{2}}, \ldots$ and therefore $\underset{n\to\infty}{\text{Lt}}\ P_nQ_n = \underset{n\to\infty}{\text{Lt}}\ 1/3^{n-1}2^{\frac{1}{2}} = 0$. Therefore, there is one and only one point P common to all segments P_1Q_1, P_2Q_2, $P_3Q_3, \ldots, P_nQ_n, \ldots$. But we have argued that there is no point of $L \cap S$ ($\equiv$ the set of points common to L and S) outside P_1Q_1, outside $P_2Q_2, \ldots,$ outside $P_nQ_n, \ldots$ Therefore it follows that there is no point of E in $L \cap S$ excepting P (if it is at all a point of E). But by Utz's (1951) process we have seen that there is at least one point of E on $L \cap S$ whenever $L \cap S$ is non-empty (i.e whenever L cuts S) and $\frac{1}{3} \leq |m| \leq 3$; and for L, i.e for $y = mx + c (\equiv y = x + \frac{1}{2})$, $m = 1$. Hence it follows that this point must be P. Hence, on the line L (whose equation is $y = x + \frac{1}{2}$), there is one and only one point $P(x, y) \varepsilon E$. Therefore it follows that $x \varepsilon C_x$ and $y \varepsilon C_y$ and $y - x = \frac{1}{2} \equiv \cdot 111111 \ldots$ ad inf.

$\therefore$ there is one and only one pair of points (x, y), such that $x \varepsilon C$, $y \varepsilon C$ and $y - x = 111111 \ldots$ ad inf., which completes the demonstration of the conclusion made above.

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A MATRIX METHOD OF REDUCTION OF THE GENERAL LORENTZ TRANSFORMATION*

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The object of this paper is to present a new method of reduction of the general Lorentz transformation in 4-space into a pair of component plane rotations. Starting with the Minkowski's representation of the general Lorentz transformation in which a complex orthogonal matrix of rank 4 is involved, it is shewn how by a matrix method this can generally be resolved into a pair of commutative orthogonal factor-matrices. A resolution by means of a pair of non-commutative factor-matrices with complex elements involving the coefficients of a unimodular spin transformation in complex space is also noted in this connection.

1. Let the general linear transformation of co-ordinates in Euclidean 4-space be defined by the set of equations

$$x'^i = a^i_j x^j, \quad (i = 1, 2, 3, 4) \tag{1.1}$$

where

$$|a^i_j| \neq 0.$$

We define (1.1) to be a general Lorentz transformation, if

$$\begin{aligned} &(a^\alpha_1)^2 + (a^\alpha_2)^2 + (a^\alpha_3)^2 - (a^\alpha_4)^2 = 1, \quad \alpha = 1, 2, 3 \\ &(a^4_1)^2 + (a^4_2)^2 + (a^4_3)^2 - (a^4_4)^2 = -1, \\ &a^\alpha_1 a^\beta_1 + a^\alpha_2 a^\beta_2 + a^\alpha_3 a^\beta_3 - a^\alpha_4 a^\beta_4 = 0, \quad \alpha \neq \beta. \end{aligned} \tag{1.2}$$

Introducing the matrix (following Minkowski)

$$A = \begin{bmatrix} a^1_1 & a^1_2 & a^1_3 & ia^1_4 \\ a^2_1 & a^2_2 & a^2_3 & ia^2_4 \\ a^3_1 & a^3_2 & a^3_3 & ia^3_4 \\ -ia^4_1 & -ia^4_2 & -ia^4_3 & a^4_4 \end{bmatrix} \tag{1.3}$$

and denoting its transposed by $\tilde{A}$, the conditions (1.2) lead to the relation

$$A\tilde{A} = \tilde{A}A = E. \tag{1.4}$$

so that A satisfies the orthogonality conditions. Further, the transformation (1.1) is represented in the matrix equation

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$$X' = AX$$

with
$$X = \bar{A}X', \tag{1.5}$$

where X represents the column-vector $(x^1, x^2, x^3, -ix^4)$ and X' the corresponding transformed vector.

2. Let Ω denote the matrix of rank 4

$$\begin{bmatrix} 0 & \omega_{12} & \omega_{13} & i\omega_{14} \\ \omega_{21} & 0 & \omega_{23} & i\omega_{24} \\ \omega_{31} & \omega_{32} & 0 & i\omega_{34} \\ i\omega_{41} & i\omega_{42} & i\omega_{43} & 0 \end{bmatrix} \quad (\omega_{ij} = -\omega_{ji}) \tag{2.1}$$

then $\bar{\Omega} = -\Omega$, and the matrix $\exp \Omega$ satisfies the condition of orthogonality (1.4). Replacing A by $\exp \Omega$ we shall study Lorentz transformation (1.5) expressed in the form

$$X' = \exp \Omega\, X = X + \Omega X + \frac{\Omega^2}{2!}X + \ldots\ldots \tag{2.2}$$

Let us write the characteristic equation satisfied by Ω in the form

$$\Omega^5 - p^2{}_1\Omega^3 - p^4{}_2\Omega = 0, \tag{2.3}$$

where
$$p^2{}_1 = -\omega^2{}_{12} - \omega^2{}_{13} - \omega^2{}_{23} + \omega^2{}_{14} + \omega^2{}_{24} + \omega^2{}_{34}, \tag{2.4}$$

$$p^4{}_2 = (\omega_{12}\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23})^2. \tag{2.5}$$

On reducing the exponents of Ω in (2.2) below 5 by means of the relation (2.3) we can express (2.2) in the form

$$X' = X + f_1(p_1, p_2)\Omega X + f_2(p_1, p_2)\Omega^2 X + f_3(p_1, p_2)\Omega^3 X + f_4(p_1, p_2)\Omega^4 X, \tag{2.6}$$

where the four functions f are to be determined.

Consider now the algebraic equation

$$x^5 - p^2{}_1 x^3 - p^4{}_2 x = 0 \tag{2.7}$$

corresponding to the characteristic equation (2.3). Let $\beta^2{}_2$ and $\beta^2{}_4$ be so determined that

$$\beta^2{}_2 - \beta^2{}_4 = p^2{}_1 \quad \text{and} \quad \beta^2{}_2\beta^2{}_4 = p^4{}_2, \tag{2.8}$$

then the roots of (2.7) may be denoted by 0, $\pm\beta_2$, $\pm i\beta_4$ and (2.3) is expressible as

$$\Omega(\Omega^2 - \beta^2{}_2)(\Omega^2 + \beta^2{}_4) = 0. \tag{2.9}$$

Now the relation
$$e^x = 1 + \sum_{l=1}^{4} f_l(p_1, p_2)x^l$$

holds good only when x satisfies the equation (2.7). We therefore, get the following identities:

$$e^{\pm\beta_2} = 1 + \beta^2{}_2 f_2 + \beta^4{}_2 f_4 \pm (\beta_2 f_1 + \beta^3{}_2 f_3),$$
$$e^{\pm i\beta_4} = 1 - \beta^2{}_4 f_2 + \beta^4{}_4 f_4 \pm i(\beta_4 f_1 - \beta^3{}_4 f_3). \tag{2.10}$$

From the above we get

$$\cosh\beta_2 = 1+\beta^2_2 f_2+\beta^4_2 f_4,\ \sinh\beta_2 = \beta_2 f_1+\beta^3_2 f_3,$$
$$\cos\beta_4 = 1-\beta^2_4 f_2+\beta^4_4 f_4,\ \sin\beta_4 = \beta_4 f_1-\beta^3_4 f_3. \tag{2.11}$$

Solving the set of equations (2.11) we obtain

$$f_1 = \frac{\beta^3_4\sinh\beta_2+\beta^3_2\sin\beta_4}{\beta_2\beta_4(\beta^2_2+\beta^2_4)},\qquad f_2 = \frac{\beta^4_2(1-\cos\beta_4)+\beta^4_4(\cosh\beta_2-1)}{\beta^2_2\beta^2_4(\beta^2_2+\beta^2_4)},$$

$$f_3 = \frac{\beta_4\sinh\beta_2-\beta_2\sin\beta_4}{\beta_2\beta_4(\beta^2_2+\beta^2_4)},\qquad f_4 = \frac{\beta^2_4(\cosh\beta_2-1)-\beta^2_2(1-\cos\beta_4)}{\beta^2_2\beta^2_4(\beta^2_2+\beta^2_4)}. \tag{2.12}$$

3. From (2.6) we see that matrix $A = \exp\Omega$ is equivalent to

$$A = E+f_1\Omega+f_2\Omega^2+f_3\Omega^3+f_4\Omega^4. \tag{3.1}$$

Utilizing the explicit expressions (2.12) for the f's we get

$$A = E+\frac{\sinh\beta_2}{\beta_2}\Gamma_2+\frac{\sin\beta_4}{\beta_4}\Gamma_4+\frac{\cosh\beta_2-1}{\beta^2_2}\Omega\Gamma_2+\frac{1-\cos\beta_4}{\beta^2_4}\Omega\Gamma_4, \tag{3.2}$$

where

$$\Gamma_2 = \frac{\beta^2_4\Omega+\Omega^3}{\beta^2_2+\beta^2_4},\ \Gamma_4 = \frac{\beta^2_2\Omega-\Omega^3}{\beta^2_2+\beta^2_4}. \tag{3.3}$$

Obviously, we have $\Gamma_2+\Gamma_4 = \Omega$ and, in view of (2.3), $\Gamma_2\Gamma_4 = 0$; consequently

$$\Omega\Gamma_2 = \Gamma^2_2,\ \Omega\Gamma_4 = \Gamma^2_4.$$

Thus (3.2) may be written in the product form

$$A = \left(E+\frac{\sinh\beta_2}{\beta_2}\Gamma_2+\frac{\cosh\beta_2-1}{\beta^2_2}\Gamma^2_2\right)\left(E+\frac{\sin\beta_4}{\beta_4}\Gamma_4+\frac{(1-\cos\beta_4)}{\beta^2_4}\Gamma^2_4\right) \tag{3.4}$$

where the order of the factors is immaterial. It may be noted that Γ_2 and Γ_4 are of type (2.1) and of rank 2 satisfying the characteristic equations

$$\Gamma^3_2 = \beta^2_2\Gamma_2,\quad \Gamma^3_4 = -\beta^2_4\Gamma_4 \tag{3.5}$$

respectively.

Observing now that the two factors in (3.4) involving Γ_2 and Γ_4 are respectively the reduced forms of e^{Γ_2} and e^{Γ_4} obtained by repeated applications of (3.5) it follows that the general Lorentz transformation is equivalent to two plane rotations performed successively.

4. We proceed now to indicate how Ω, with Γ_2 and Γ_4, can be obtained when the matrix A is given.

From (3.2) we have (A, not symmetric)

$$\frac{A-\bar{A}}{2} = \frac{\sinh\beta_2}{\beta_2}\Gamma_2+\frac{\sin\beta_4}{\beta_4}\Gamma_4. \tag{4.1}$$

Denoting $(A-\bar{A})/2$ by Φ, which is of type (2.1), it may be seen that Φ satisfies the characteristic equation

$$\Phi^5+\Phi^3(\sin^2\beta_4-\sinh^2\beta_2)-\Phi\sin^2\beta_4\sinh^2\beta_2 = 0, \tag{4.2}$$

that is;

$$\Phi(\Phi^2-\sinh^2\beta_2)(\Phi^2+\sin^2\beta_4) = 0.$$

Referring to (2.9) we notice that the characteristic roots of Ω and Φ are related. Corresponding to the real roots $\pm\beta_2$ of Ω, we have $\pm\sinh\beta_2$ of Φ and corresponding to the pair $\pm i\beta_4$ of Ω we have the pair of imaginary roots $\pm i\sin\beta_4$ of Φ. The matrix Ω can, however, be expressed by means of the determinantal equation

$$\begin{vmatrix} \Omega & \beta_2 & \beta_4 \\ \Phi & \sinh\beta_2 & \sin\beta_4 \\ \Phi^3 & \sinh^3\beta_2 & -\sin^3\beta_4 \end{vmatrix} = 0. \tag{4.3}$$

To obtain Γ_2 and Γ_4 in terms of Φ we first express Ω^3 by means of the determinantal equation

$$\begin{vmatrix} \Omega^3 & \beta^3_2 & -\beta^3_4 \\ \Phi & \sinh\beta_2 & \sin\beta_4 \\ \Phi^3 & \sinh^3\beta_2 & -\sin^3\beta_4 \end{vmatrix} = 0. \tag{4.4}$$

Making use of the relations (3.3) we then readily obtain

$$\begin{vmatrix} \Gamma_2 & \beta_2 & 0 \\ \Phi & \sinh\beta_2 & \sin\beta_4 \\ \Phi^3 & \sinh^3\beta_2 & -\sin^3\beta_4 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} \Gamma_4 & 0 & \beta_4 \\ \Phi & \sinh\beta_2 & \sin\beta_4 \\ \Phi^3 & \sinh^3\beta_2 & -\sin^3\beta_4 \end{vmatrix} = 0, \tag{4.5}$$

which express Γ_2 and Γ_4 in terms of Φ.

5. The above treatment is much simplified in the case of special Lorentz transformation. We observe from (2.3) that if $p^4_2 = 0$, Ω is of rank 2, satisfying the characteristic equation

$$\Omega^3-p^2_1\Omega = 0. \quad (p^2_1 \text{ positive}) \tag{5.1}$$

The matrix $A = \exp\Omega$ now corresponds to the special Lorentz transformation. On reducing the exponents of Ω in $\exp\Omega$ below 3 by means of (5.1) we get

$$A = E+\frac{\sinh p_1}{p_1}\Omega+\frac{\cosh p_1-1}{p_1^2}\Omega^2, \tag{5.2}$$

which represents a plane rotation with imaginary angle ip_1. From (5.2) we have

$$\frac{A-\bar{A}}{2} = \Phi = \frac{\sinh p_1}{p_1}\Omega. \tag{5.3}$$

Further, it is found that Φ satisfies the characteristic equation

$$\Phi^3-\sinh^2 p_1\,\Phi = 0. \tag{5.4}$$

Thus when A is given the terms in (5.2) are all determined by means of the auxiliary matrix Φ.

In case p^2_1 be negative, say $-\beta^2_4$, the reduced form of exp Ω will be

$$A = E + \frac{\sin \beta_4}{\beta_4}\Omega + \frac{1-\cos \beta_4}{\beta^2_4}\Omega^2, \tag{5.5}$$

Ω satisfying the characteristic equation

$$\Omega^3 + \beta^2_4\Omega = 0. \tag{5.6}$$

6. We make now the following observations regarding the reduction of A when it is orthogonal and symmetric. Since $A = \bar{A}$ it follows from (3.1) that

$$f_1\Omega + f_3\Omega^3 = 0, \tag{6.1}$$

so that Ω is of rank 2. It is further noted that the characteristic equation satisfied by Ω will be either

$$(i) \qquad \Omega^3 - \beta^2_2\Omega = 0, \qquad \text{or} \quad (ii) \quad \Omega^3 + \beta^2_4\Omega = 0.$$

In the first case the reduced form of A will be

$$A = E + \frac{\sinh \beta_2}{\beta_2}\Omega + \frac{\cosh \beta_2 - 1}{\beta^2_2}\Omega^2 \tag{6.2}$$

with

$$\frac{\sinh \beta_2}{\beta_2} = 0 \text{ and } \cosh \beta_2 = -1. \tag{6.3}$$

In the second case the reduced form of A will be

$$A = E + \frac{\sin \beta_4}{\beta_4}\Omega + \frac{1-\cos \beta_4}{\beta^2_4}\Omega^2 \tag{6.4}$$

with

$$\frac{\sin \beta_4}{\beta_4} = 0 \text{ and } \cos \beta_4 = -1. \tag{6.5}$$

If $\beta_4 = \pi$, the conditions (6.5) are satisfied and (6.4) reduces to

$$A = E + \frac{2}{\pi^2}\Omega^2. \tag{6.6}$$

In the first case no real β_2 satisfies (6.3).

An interesting algebraic method of decomposition is given by Bose (1939).

7. Next we pass on to the resolution of A in (1.3) into a pair of non-commutative factor-matrices. Modifying Whittaker's formula (1937) we can write

$$A = M.N^*,$$

where

$$M = \frac{1}{\sqrt{2}}\begin{bmatrix} \gamma & \delta & \alpha & \beta \\ -i\gamma & -i\delta & i\alpha & i\beta \\ \alpha & \beta & -\gamma & -\delta \\ -i\alpha & -i\beta & -i\gamma & -i\delta \end{bmatrix}, \quad N^* = \frac{1}{\sqrt{2}}\begin{bmatrix} \beta^* & -i\beta^* & \alpha^* & i\alpha^* \\ \alpha^* & i\alpha^* & -\beta^* & i\beta^* \\ \delta^* & -i\delta^* & \gamma^* & i\gamma^* \\ \gamma^* & i\gamma^* & -\delta^* & i\delta^* \end{bmatrix}, \tag{7.1}$$

α, β, γ, δ being complex parameters satisfying the relation

$$\alpha\delta - \beta\gamma = 1. \tag{7.2}$$

In the above the asterisk denotes the complex conjugate.

Consider now the spin coordinate transformation in a two-dimensional complex space defined by

$$x'^1 = f^1(x^1, x^2),$$
$$x'^2 = f^2(x^1, x^2), \tag{7.3}$$

where $x' = x_0 + ix_3$, $x^2 = x_1 + ix_2$, x_i's being real.

Further, let us impose the condition that the Jacobian

$$\frac{\partial(f^1, f^2)}{\partial(x^1, x^2)} = 1, \tag{7.4}$$

which corresponds to a unimodular spin transformation. Then α, β, γ, δ in (7.1) may be identified with the 4 coefficients of transformation in (7.3), namely;

$$\alpha = \frac{\partial x'^1}{\partial x^1}, \; \beta = \frac{\partial x'^1}{\partial x^2}, \; \gamma = \frac{\partial x'^2}{\partial x^1}, \; \delta = \frac{\partial x'^2}{\partial x^2}. \tag{7.5}$$

The 8 basic matrices involved in (7.1) have many interesting properties. These may be used in constructing the set of 16 Eddington matrices of order four.

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EXACT SOLUTION OF THE BURIED LINE SOURCE PROBLEM FOR A UNIFORMLY MOVING LINE SOURCE

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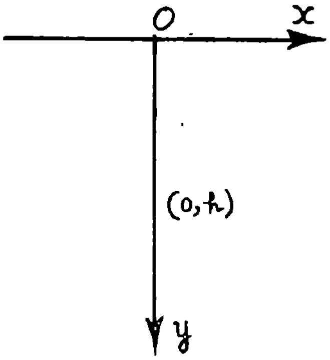
Introduction. Nakano (1925), Lapwood (1949), and Garvin (1956) have considered the motion produced in a semi-infinite, elastic, homogeneous, isotropic medium by the sudden creation at a certain depth below the free surface of a dilatational line source. In this paper the problem is considered with the change that the source begins to move, immediately after creation, parallel to the surface with constant velocity. The wave function representing the source is obtained from the corresponding differential equation and the method of Cagniard (1939) is used to obtain the surface displacement in exact terms. The motions of similar moving sources in an infinite medium have been considered by Sneddon (1956) and others.

Problem. The line source of constant strength a is assumed to be created suddenly at $t = 0$ at $(0, h)$ in a semi-infinite, homogeneous isotropic elastic medium lying in the region $y \geq 0$, and to move immedietely with constant velocity $V (< \alpha)$ parallel to the x-axis.



The wave function φ_0 determined by the source is assumed to satisfy the differential equation—

$$\left.\begin{aligned} \nabla^2\varphi_0 - \frac{1}{\alpha^2}\frac{\partial^2\varphi_0}{\partial t^2} &= 0, \quad t < 0 \\ &= -\pi a\delta(y-h)\delta(x-Vt), \quad t \geq 0, \end{aligned}\right\} \tag{1}$$

where α is the compressional wave velocity in the medium and δ denotes Dirac's delta function. The term on the right hand side of (1) for $t \geq 0$ indicates the presence of the source.

The displacement components at the free surface in the presence of the source will be obtained under the following conditions: (1) the stress components vanish at $y = 0$, (2) the displacement is bounded at infinity.

$$\bar{f}(x, y, p) \quad (p \text{ being positive}),$$

the Laplace transform of $f(x, y, t)$, is given by the relation

$$\bar{f}(x, y, p) = \int_0^\infty e^{-pt} f(x, y, t)dt.$$

Transformed quantites being denoted by superimposed bars, (1) transforms to

$$\left.\begin{aligned} \nabla^2\bar{\varphi}_0 - \frac{p^2}{\alpha^2}\bar{\varphi}_0 &= 0, \quad x < 0 \\ &= -\pi a/V\delta(y-h)e^{-px/V}, \quad x \geq 0 \end{aligned}\right\} \tag{2}$$

since the medium is initially at rest.

Using $$\bar{\varphi}(x, y, p) = \int_0^\infty e^{i\eta y} A(x, \eta, p)d\eta$$

the following equation is obtained from (2):

$$\left.\begin{aligned} \frac{\partial^2 A}{\partial x^2} - \left(\frac{p^2}{\alpha^2} + \eta^2\right)A &= 0, \quad x < 0 \\ &= -a/2Ve^{-px/V - i\eta h}, \quad x \geq 0 \end{aligned}\right\} \tag{3}$$

The solution of (3) which is bounded for all values of x contains two arbitrary constants. Since A is a bounded function of x both A and $\partial A/\partial x$ are continuous at $x = 0$. These conditions determine the two cosntants.

$\bar{\varphi}_0$ is then given by the equations

$$\begin{aligned} \bar{\varphi}_0 &= \frac{a}{2V}\int_{-\infty}^{+\infty} \frac{e^{i\eta(y-h)+\eta'_p x}}{p^2/V^2 - p^2/\alpha^2 - \eta^2}\left\{\frac{p/V}{\eta'_p} - 1\right\}d\eta, \quad x < 0 \\ &= \frac{a}{2V}\int_{-\infty}^{+\infty} \frac{e^{i\eta(y-h)}}{p^2/V^2 - p^2/\alpha^2 - \eta^2}\left[e^{-\eta'_p x}\left\{\frac{p/V}{\eta'_p} + 1\right\} - 2e^{-px/V}\right]d\eta, \quad x \geq 0 \\ &= \frac{a}{8\pi V}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{e^{-i\eta(y-h)+i\xi x}}{(p^2/\alpha^2 + \eta^2 + \xi^2)(p/V + i\xi)}\, d\xi d\eta, \text{ where } \eta'_p = \left(\frac{p^2}{\alpha^2} + \eta^2\right)^{\frac{1}{2}} \\ &= \frac{a}{4}\int_{-\infty}^{+\infty} \frac{e^{-i\xi x - |y-h|\eta_p}}{\eta_p(p + i\xi V)}d\xi, \end{aligned}$$

on interchanging the integrations and integrating with respect to η.

Hence $$\varphi_0(x, y, t) = \frac{a\alpha}{2}\int_0^\infty \frac{e^{-(|y-h|/\alpha)(p^2+\alpha^2\xi^2)^{\frac{1}{2}}}}{(p^2+\alpha^2\xi^2)^{\frac{1}{2}}} \cdot \frac{p\cos\xi x+\xi V\sin\xi x}{p^2+\xi^2V^2}\,d\xi,$$

where $$\eta_p = (p^2/\alpha^2+\xi^2)^{\frac{1}{2}}$$

Using the notation $\bar{f}(x, y, p)\subset f(x, y, t)$ we have

$$\frac{e^{-(|y-h|/\alpha)(p^2+\alpha^2\xi^2)^{\frac{1}{2}}}}{(p^2+\alpha^2\xi^2)^{\frac{1}{2}}}\subset\begin{cases} 0, & 0<t<|y-h|/\alpha \\ J_0(\xi(\alpha^2t^2-(y-h)^2)^{\frac{1}{2}}), & |y-h|/\alpha<t. \end{cases} \qquad \text{(Erdelyi)}$$

$$\frac{p}{p^2+\xi^2V^2}\subset\cos V\xi t$$

$$\frac{\xi V}{p^2+\xi^2V^2}\subset\sin V\xi t,$$

Hence by the product theorem for Laplace transforms

$$\varphi_0(x, y, t) = 0, \qquad 0<t<|y-h|/\alpha$$

$$= \frac{a\alpha}{2}\int_0^\infty d\xi\int_{|y-h|/\alpha}^t J_0(\xi(\alpha^2\lambda^2-(y-h)^2)^{\frac{1}{2}})\cos\xi(Vt-x-V\lambda)d\lambda, |y-h|/\alpha<t.$$

The integrations with respect to ξ and λ may be shown to be interchangeable.

Now $$I = \int_0^\infty J_0(\xi(\alpha^2\lambda^2-(y-h)^2)^{\frac{1}{2}})\cos\xi(Vt-x-V\lambda)d\lambda,$$

$$= 0, \qquad \text{or} \qquad 1/(F(\lambda))^{\frac{1}{2}}$$

according as $F(\lambda) = \alpha^2\lambda^2-(y-h)^2-(Vt-x-V\lambda)^2$ is positive or negative.

$F(\lambda)$ is always $-$ve for $|y-h|/\alpha<\lambda<t$ if $t<(x^2+(y-h)^2)^{\frac{1}{2}}/\alpha$ and it is + ve for $\lambda_1<\lambda<t$ where λ_1 is a quantity lying between $|y-h|/\alpha$ and t if $t>(x^2+(y-h)^2)^{\frac{1}{2}}/\alpha$.

Therefore $$\varphi_0(x, y, t) = 0, \qquad 0<t<\frac{(x^2+(y-h)^2)^{\frac{1}{2}}}{\alpha}$$

$$= \frac{\alpha a}{2}\int_{\lambda_1}^t \frac{\alpha\lambda}{(F(\lambda))^{\frac{1}{2}}}, \qquad \frac{(x^2+(y-h)^2)^{\frac{1}{2}}}{\alpha}<t.$$

The latter integral is easily obtained as

$$\frac{1}{(\alpha^2-V^2)^{\frac{1}{2}}}\cosh^{-1}\left[\frac{\alpha^2t-Vx}{\{\alpha^2(Vt-x)^2+(\alpha^2-V^2)(y-h)^2\}^{\frac{1}{2}}}\right]$$

This expression was obtained by Walters (1951) in a different form.

If $V = 0$, $$\varphi_0(x, y, t) = 0, \qquad 0<t<\frac{(x^2+(y-h)^2)^{\frac{1}{2}}}{\alpha},$$

$$= \frac{a}{2}\cosh^{-1}\frac{\alpha t}{(x^2+(y-h)^2)^{\frac{1}{2}}}, \qquad \frac{(x^2+(y-h)^2)^{\frac{1}{2}}}{\alpha}<t,$$

This function is identical with that used by Garvin.

The x-and y-components of the displacement in the medium are given by

$$\left.\begin{aligned} u &= -\frac{\partial\varphi}{\partial x}+\frac{\partial\psi}{\partial y} \\ v &= -\frac{\partial\varphi}{\partial y}-\frac{\partial\psi}{\partial x} \end{aligned}\right\}$$

where $\varphi = \varphi_0+\varphi_1, \qquad \psi = \psi_1,$

φ_1 and ψ_1 satisfying the wave equations (β = shear wave velocity in the medium),

$$\nabla^2\varphi_1 = \frac{1}{\alpha^2}\cdot\frac{\partial^2\varphi_1}{\partial t^2}$$

$$\nabla^2\psi_1 = \frac{1}{\beta^2}\cdot\frac{\partial^2\psi_1}{\partial t^2}$$

The stress components $\widehat{xy}$, $\widehat{yy}$ are given by

$$\left.\begin{aligned} \widehat{xy} &= \mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \\ \widehat{yy} &= \varrho\left[(\alpha^2-2\beta^2)\frac{\partial u}{\partial x}+\alpha^2\frac{\partial v}{\partial y}\right] \end{aligned}\right\}$$

where μ, ϱ are the coefficients of rigidity and density respectively of the medium.

The solutions of the Laplace transforms of the above equations for φ_1, ψ_1 consistent with (1), (2), are assumed to be

$$\bar{\varphi}_1 = \frac{a}{2}\int_0^\infty e^{-y\eta_\beta}[E_1\cos\xi x+E_2\sin\xi x]d\xi$$

$$\bar{\psi}_1 = \frac{a}{2}\int_0^\infty e^{-y\eta_\beta}[F_1\cos\xi x+F_2\sin\xi x]d\xi.$$

where $\eta_\beta = (p^2/\beta^2+\xi^2)^{\frac{1}{2}}.$

Since $\widehat{xy}$, $\widehat{yy}$ vanish at $y = 0$, their Laplace transforms will also vanish at $y = 0$. These two conditions give the relations

$$2\beta^2\xi\eta_p E_1-(p^2+2\beta^2\xi^2)F_2 = \frac{2\xi\beta^2 pe^{-\eta_\beta h}}{p^2+\xi^2V^2}$$

$$2\beta^2\xi\eta_p E_2+(p^2+2\beta^2\xi^2)F_1 = \frac{2\beta^2\xi^2 Ve^{-\eta_\beta h}}{p^2+\xi^2V^2}$$

$$(p^2+2\beta^2\xi^2E_1-2\beta^2\xi\eta sF_2 = \frac{-p(p^2+2\beta^2\xi^2)}{\eta_p(p^2+\xi^2V_2)}e^{-\eta_p h}$$

$$(p^2+2\beta^2\xi^2)E_2+2\beta^2\eta sF_1 = \frac{-\xi V(p^2+2\beta^2\xi^2)}{\eta_p(p^2+\xi^2V^2)}e^{-\eta_p h}$$

Solving these equations and using the values of E_1, E_2, F_1, F_2 the Laplace transforms of the displacement components at $(x, 0)$ are obtained to be

$$\frac{\bar{u}(x, 0, p)}{a} = -2\beta^2p^2 \ \mathbf{Re}\int_0^\infty \frac{\xi^2V\eta_s e^{-i\xi x-h\eta_p}}{\Delta(p, \xi)(p^2+\xi^2V)}\,d\xi-2\beta^2p^2 \ \mathbf{Im}\int_0^\infty \frac{p\xi\eta_s e^{-i\xi x-h\eta_p}}{\Delta(p, \xi)(p^2+\xi^2V^2)}\,d\xi$$

$$\frac{\bar{v}(x, 0, p)}{a} = \frac{2\bar{v}_0(x, 0, p)}{a}+2\beta^2p \ \mathbf{Re}\int_0^\infty \frac{\xi^2e^{-h\eta_p-i\xi x}(p^2 2\beta^2\xi^2-2\beta^2\eta_s\eta_p)d\xi}{\Delta(p, \xi)(p^2+\xi^2V^2)}$$

$$-2\beta^2 \ \mathbf{Im}\int_0^\infty \frac{\xi^3Ve^{-h\eta_p-i\xi x}(p^2+2\beta^2\xi^2-2\beta^2\eta_s\eta_p)}{\Delta(p, \xi)(p^2+\xi^2V^2)}\,d\xi$$

where $$\Delta(p, \xi) = (p^2+2\beta^2\xi^2)^2-4\beta^4\xi^2\eta_s\eta_p.$$

and $v_0(x, 0, t)$ is the y-component of the displacement at $(x, 0)$ at time t due to the source alone.

On putting $\nu = \frac{\beta}{\alpha}$, $\nu' = \frac{V}{\alpha}$, $z = \frac{\xi\alpha}{p}$, $t = \frac{h(1+z^2)^{\frac{1}{2}}+izx}{\alpha}$, the above expressions may be written

$$\left.\begin{aligned}\frac{\bar{u}(x, 0, p)}{a} &= -\frac{2\nu\nu'}{\alpha}\ \mathbf{Re}\int_0^\infty zF_1(\nu, z)e^{-pt}dz-\frac{2\nu}{\alpha}\ \mathbf{Im}\int_0^\infty F_1(\nu, z)e^{-pt}dz.\\ \frac{\bar{v}(x, 0, p)}{\alpha} &= \frac{2\bar{v}_0(x, 0, p)}{\alpha}+\frac{2\nu^2}{\alpha}\ \mathbf{Re}\int_0^\infty F_2(\nu, z)e^{-pt}dz-\frac{2\nu^2\nu'}{\alpha}\ \mathbf{Im}\int_0^\infty zF_2(\nu, z)e^{-pt}dz\end{aligned}\right\}$$

where $$F_1(\nu, z) = \frac{z(1+\nu^2z^2)}{\Delta_0(\nu, z)}$$

$$F_2(\nu, z) = [(1+2\nu^2z^2)-2\nu(1+z^2)^{\frac{1}{2}}(1+\nu^2z^2)^{\frac{1}{2}}]/\Delta_0(\nu, z)$$

$$\Delta_0(\nu, z) = (1+\nu'^2z^2)[(1+2\nu^2z^2)^2-4\nu^3z^2(1+z^2)^{\frac{1}{2}}(1+\nu^2z^2)^{\frac{1}{2}}]$$

$\Delta_0(\nu, z)$ has simple zeroes at $z=\pm i\alpha/V_R$, $\pm i/\nu'$, V_R being the Rayleigh wave velocity in the medium; $(1+z^2)^{\frac{1}{2}}$, $(1+\nu^2z^2)^{\frac{1}{2}}$ have branch points at $z=\pm i$, $\pm i/\nu$. It will be assumed here that V is less than β

Garvin determined the nature of the mapping of the z-plane onto the t-plane by the transformation

$$t = \frac{h}{\alpha}(1+z^2)^{\frac{1}{2}}+\frac{izx}{\alpha}.$$

It is easily shown by the method of Garvin that

$$\frac{u(x,0,t)}{a} = 0, \qquad 0<t<\frac{(x^2+h^2)^{\frac{1}{2}}}{\alpha},$$

$$= -\frac{2\nu\nu'}{\alpha}\,\mathbf{Re}\left[z(t)F_1\{\nu, z(t)\}\frac{dz(t)}{dt}\right]-\frac{2\nu}{\alpha}\,\mathbf{Im}\left[F_1\{\nu, z(t)\}\frac{dz(t)}{dt}\right], \quad \frac{(x^2+h^2)^{\frac{1}{2}}}{\alpha}<t,$$

$$\frac{v(x,0,t)}{a} = 0, \qquad 0<t<\frac{(x^2+h^2)^{\frac{1}{2}}}{\alpha}$$

$$=\frac{2v_0(x,0,t)}{a} + \frac{2\nu^2}{\alpha}\,\mathbf{Re}\left[F_2\{\nu, z(t)\}\frac{dz(t)}{dt}\right]$$

$$-\frac{2\nu^2\nu'}{\alpha}\,\mathbf{Im}\left[z(t), F_2\{\nu, z(t)\}\frac{dz(t)}{dt}\right], \quad \frac{(x^2+h^2)^{\frac{1}{2}}}{\alpha}>t,$$

where $$z(t) = \frac{\alpha}{h^2+x^2}\left[-\iota xt+h\left(t^2-\frac{h^2+x^2}{\alpha^2}\right)^{\frac{1}{2}}\right]$$

and $v_0(x, 0, t) = 0,$ $$0<t<\frac{(x^2+h^2)^{\frac{1}{2}}}{\alpha}$$

$$= \frac{a\alpha(\alpha^2-V^2)^{\frac{1}{2}}}{2}\cdot\frac{h(Vx-\alpha^2t)}{\{(Vx-\alpha^2t)^2-\alpha^2(Vt-x)^2-(\alpha^2-V^2)h^2\}^{\frac{1}{2}}[\alpha^2(Vt-x)^2+(\alpha^2-V^2)h^2]},$$

$$\frac{(x^2+h^2)^{\frac{1}{2}}}{\alpha}<t.$$

The displacement components have been numerically calculated in the particular case $V = \alpha/2$, $x = 2h$, $\nu = 1/\sqrt{3}$ for a number of values of t. The results are given in the table below. u, v both oscillate once and then decrease steadily.

$t=\frac{t\alpha}{h}$	2·5	3	4	5	6	7	8	9	10
$\frac{hu}{a}$	−1·82	1·08	·76	−·47	−·75	−·78	−·63	−·54	−·47
$\frac{hv}{a}$	−1·23	−1·09	−1·86	−·92	−·45	−·26	−·17	−·12	−·09

Remark. By superposing the solution due to a negative source starting at a slightly later time at a different point, a solution is obtained which may represent an explosive disturbance, the point at which the disturbance begins being different from that at which it ends. Also the partial derivatives with respect to x of the above expressions for u, v give the displacement components for a moving dipole in two dimensions, oriented parallel to the x-axis.

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ROTATORY FLOW OF VISCOPLASTIC MATERIAL OF BINGHAM TYPE

I. FLOW BETWEEN COAXIAL CIRCULAR CYLINDERS

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Summary. Rotatory flow of viscoplastic material of Bingham type between two coaxial circular cylinders has been investigated for the constitutive equations developed by Oldroyd. For steady flow both 'free' and 'plug' motions have been discussed while in unsteady flow only free motion has been considered. Particular emphasis has been put on determining the critical value of the Bingham number above or below which the flow pattern is a possible one.

1. **Introduction.** Since Bingham and Green (1919) discovered that oil paint (before drying) is a plastic solid and not a viscous liquid, attempts have been made to explain the actual mechanical behaviour of similar material by combining linearly the laws governing the Newtonian viscous liquid and the perfectly plastic Mises solid. Following Bingham (1922) and Houwink (1937), a material which can support a finite stress elastically without flow and which flows with constant mobility (or plastic fluidity) when the stresses are sufficiently great, is called a viscoplastic solid of Bingham type. The mathematical formulation of the equations of flow for this solid has been given by Oldroyd (1947a); an account may also be found in works Reiner (Reiner, 1943, Eirich, 1956) and Prager (Eirich, 1956). As illustrations of the general theory, special problems solved so far either exactly or approximately include the steady rectilinear flow (*i*) through a circular pipe by Buckingham (1921) and Reiner (1926), (*ii*) between two parallel planes by Prager (Eirich, 1956) and (*iii*) between co-axial rotating circular cylinders by Reiner-Rivlin (Eirich, 1956). In a series of papers Oldroyd (1947b, 1947c, 1948a, 1948b) has considered steady rectilinear flow between co-axial cylinders, eccentric circular cylinders, confocal elliptic cylinders and cylinders of arbitrary cross-section as well as unsteady rectilinear flow between parallel planes and through a circular pipe. He (Oldroyd, 1947d) has also introduced the concept of plastic boundary layer and Bingham number in analogy with Prandtl viscous boundary layer and Reynolds number respectively. The only uniqueness theorem concerning stress and velocity fields is due to Prager (1954) Ilyushin (1940) and Ishlinky (1943) have discussed several problems regarding the stability of viscoplastic flow, an account of which may be found in the work of Nadai (1950). Recently Sliber and Paslay (1957) has solved a problem of flow between coaxial cylinders. In the present paper the rotatory flow both steady and unsteady, between coaxial circular cylinders has been investigated. The inner cylinder is supposed to be at rest while the flow is caused by the rotation of the outer one. Though the characteristic feature for the steady motion is known already its fresh derivation and discussion with reference to

Bingham number clarifies ideas that follow. The steady flow occurs throughout the whole region if the Bingham number is less than a critical value while for its greater values flow region and elastic region (plug motion) occur side by side. In the unsteady flow we have considered the case when the whole region is in motion. It is found that for a prescribed Bingham number such a motion is possible only for a particular interval of time. A discussion of plug motion in unsteady flow will be given in a subsequent paper.

2. Fundamental equations. The rheological equations of state are (Reiner, 1943; Oldroyd, 1947a)

$$p_{ii} = 3k\Delta \quad \text{(in elastic regions only,} \quad \tfrac{1}{2}p'_{ik}p'_{ik} \leq \odot^2) \tag{1}$$

$$p_{ik} = 2\mu\epsilon'_{ik} \quad \text{(in elastic regions only,} \quad \tfrac{1}{2}p'_{ik}p'_{ik} \leq \odot^2) \tag{2}$$

$$p'_{ik} = 2\eta e'_{ik} \quad \text{(in flow regions only,} \quad \tfrac{1}{2}p'_{ik}p'_{ik} \geq \odot^2) \tag{3}$$

with

$$\eta = \eta_1 + \odot(2e'_{ik}e'_{ik})^{-\frac{1}{2}}. \tag{4}$$

In the above equations, η_1 is the (constant) reciprocal mobility, v is the (constant) yield value, μ is the (constant) rigidity modulus in the elastic region and k (not necessarily constant) is the bulk modulus whereas ϵ_{ij} is the strain tensor, e_{ij} is the rate of strain tensor, $\Delta = \epsilon_{ii}$ is the dilatation, p_{ij} is the stress tensor and primes denote deviatoric components of tensors, *e.g.*,

$$p'_{ik} = p_{ik} + p\delta_{ik}, \qquad p = -\tfrac{1}{3}p_{ii} \tag{5}$$

δ_{ik} being the substitution tensor. On the yield surface, the surface of separation between elastic regions and plastic flow regions, the transition conditions are that e'_{ik} must vanish identically and the velocity must be continuous, the elastic region being treated as rigid. The equations of motion and continuity are

$$\varrho\frac{D\odot_i}{Dt} = \frac{\partial p'_{ik}}{\partial x_k} - \frac{\partial p}{\partial x_i} + \varrho X_i \tag{6}$$

$$\frac{D\varrho}{Dt} + \varrho e_{ii} = 0 \tag{7}$$

where $\odot_i$ is the velocity vector, X_i is the external force vector, ϱ is the density and D/Dt denotes differentiation with respect to time following the material particle. We also have

$$e_{ij} = \tfrac{1}{2}(\odot_{i,j} + \odot_{j,i}) \tag{8}$$

The above equations with appropriate boundary conditions on stresses, velocity and pressure determine the velocity field as well as the yield surface as a part of the solution.

3. Steady flow. We assume the velocity components in the cylindrical coordinates (r, θ, z) to be

$$\odot_1 = \odot_r = 0, \qquad \odot_3 = \odot_z = 0 \qquad \text{and} \quad \odot_2 = \odot_\theta = u(r)$$

where u is a function of r only. From (8), (4) and (8) the nonvanishing components of strain-rate and stress tensors are

$$e'_{r\theta} = e_{r\theta} = \tfrac{1}{2}\left(\frac{du}{dr} - \frac{u}{r}\right) \tag{9}$$

$$p'_{r\theta} = p_{r\theta} = 2\eta_1 e_{r\theta} + (\odot e_{r\theta})/|e_{r\theta}| \tag{10}$$

If $e_{r\theta}$ is assumed to be positive everywhere in the flow region, the latter equation can be written as

$$p'_{r\theta} = p_{r\theta} = 2\eta_1 e_{r\theta} + \odot. \tag{10}$$

The equation of continuity (7) is satisfied if ϱ is taken as a constant while the equations of motion (6) in the absence of body forces reduce to

$$(\varrho u^2)/r = \frac{\partial p}{\partial r},\ \frac{dp_{r\theta}}{dr} + (2/r)p_{r\theta} - (1/r)\frac{\partial p}{\partial \theta} = 0,\ \frac{\partial p}{\partial z} = 0 \tag{11}$$

which show that p can be taken as a function of r only. Substituting (10) in the second relation of (11) and integrating we get

$$2\eta_1 e_{r\theta} + \odot = 2C/r^2 \tag{12}$$

where C is a constant. Again, substitution of (9) in (12) and then integration give

$$\eta_1 u = Dr - (C/r) - \odot r \log r \tag{13}$$

where D is another constant. If flows occur throughout the whole region $r_i < r < r_e$ the boundary conditions $u = 0$ on $r = r_i$ and $u = U$ on $r = r_e$ determine C and D as

$$\left.\begin{aligned} C &= a\eta_1 U r_e(a - B\log a)/(1-a^2) \\ D &= \eta_1 U[1 - Ba\log a + (B/a)(1-a^2)\log r_e]/[r_e(1-a^2)] \end{aligned}\right\} \tag{14}$$

where $B = (\odot r_i)/(\eta_1 U)$ is the dimensionless Bingham number and $a = r_i/r_e$ Relations (13) and (14) imply

$$\begin{aligned}(u/U) = \{(r/r_e)(1 - Ba\log a)/(1-a^2)\} - \{[(ar_e)/r](a - B\log a)/(1-a^2)\} \\ + \{[(Br)/(ar_e)]\log(r_e/r)\}\end{aligned} \tag{15}$$

The relation (12) now reduces to

$$e_{r\theta} = \{aUr_e(a - B\log a)/[r^2(1-a^2)]\} - \{(BU)/(2ar_e)\}. \tag{16}$$

In order to satisfy the condition of flow viz. $e_{r\theta} > 0$, in the whole region $r_i < r < r_e$, we replace r by r_e in (16) and thus get a critical value $B = B_1$ such that

$$B < B_1 = \{(2a^3)/(1 - a^2 + 2a^2\log a)\}. \tag{17}$$

If $B > B_1$, the flow is confined in a region $r_i < r < R(R < r_e)$ and the region $R < r < r_e$ moves as a rigid body. In this case the boundary conditions are $u = 0$ on $r = r_i$ and $u = (UR)/r_e$ on $r = R$ while relations (14), (15) and (16) are modified as

$$\left.\begin{aligned} C &= a\eta_1 U r_e (a - B \log a_1)/(1 - a_1^2) \\ D &= \eta_1 U[1 + (B/a)(\log R - a_1^2 \log r_i)]/[r_e(1 - a_1^2)] \end{aligned}\right\} \quad (14a)$$

$$(u/U) = \{(r/r_e)[1 - (Ba_1^2/a) \log a + (B/a) \log (a/a_1)]/(1 - a_1^2)\} - \{(r_e/r)a(a - B \log a_1)/(1 - a_1^2)\} + \{[(Br)/(ar_e)] \log (r_e/r)\} \quad (15a)$$

$$e_{r\theta} = \{aUr_e(a - B \log a_1)/[r^2(1 - a_1^2)]\} - \{(BU)/(2ar_e)\} \quad (16a)$$

where $$a_1 = r_i/R.$$

To satisfy the conditoin $e_{r\theta} > 0$ for $r_i < r < R$ we replace r by R in (16a) and get a critical value $B = B_2$ such that

$$B < B_2 = (2a\,a_1^2)/(1 - a_1^2 + 2a_1^2 \log a_1) \quad (17a)$$

It is to be noticed that 'a_1' lies between the least value 'a' and the greatest value unity (since $r_i < R < r_e$) and $B = (2aa_1^2)/(1 - a_1^2 + 2a_1^2 \log a_1)$ is an increasing function of 'a_1'. The value of 'B' increases from the least value 'B_1' (corresponding to $a_1 = a$) to infinity (corresponding to $a_1 = 1$) Conversely, 'a_1' may be defined as the increasing function of 'B' and 'a_1' tends to zero as 'B' tends to zero while the line $a_1 = 1$ serves as an asymptote for large values of 'B'. A Newtonian liquid is a limiting case of a Bingham solid with $B = 0(\odot = 0)$ and consequently $a_1 = o$ *i.e.* R infinite. As 'B' increases from zero the possible region of flow contracts gradually, and when a critical value 'B_1' is passed, flow in the whole region between the cylinders ceases to be possible. A plug of elastic solid then forms at the outer boundary; the region of flow contracts as B increases further and is confined to a thin sheath surrounding the inner cylinder for very large values of 'B'. However, for finite values of 'B' (greater than 'B_1'), R is finite *i.e.* there is a marked yield surface. Table I gives values of 'B' for different values of 'a_1' when 'a' is prescribed. Since 'a_1' is equal to 'a' for flow throughout the whole region $r_i < r < r_e$, the numbers written diagonally in Table I correspond to the critical value $B = B_1$ and these can also be verified from the result (17).

TABLE I.

Values of 'B' as a function of 'a' and 'a_1'.

	0.2	0.4	0.5	0.6	0.8	0.9	1.0
0.2	0.0192	—	—	—	—	—	—
0.4	0.1171	0.2342	—	—	—	—	—
0.5	0.2479	0.4958	0.6197	—	—	—	—
0.6	0.5290	1.0580	1.3221	1.5870	—	—	—
0.8	3.4453	6.8904	8.6133	10.3359	13.7812	—	—
0.9	15.7480	31.4960	39.3701	47.2441	62.9921	70.8662	—
1.0				infinity			

4. Unsteady flow. The velocity components are taken to be $\dot{\odot}_1 = \dot{\odot}_r = 0$, $\odot_3 = \odot_z = 0$ and $\odot_2 = \odot_\theta = u(r, t)$, where u is now a function of r and t. The non-vanishing components of rate of strain and stress are still given by (9) and (10). The equation of continuity is satisfied by taking ϱ as a constant. If $p(r, t)$ is assumed to be a function of r and t only, the third equation of motion is identically satisfied, while the first two reduce to

$$(\varrho u^2)/r = \frac{\partial p}{\partial r},\ \varrho\frac{\partial u}{\partial t} = \frac{\partial p_{r\theta}}{\partial r} + (2/r)p_{r\theta}$$

When expressed in terms of velocity component with the help of (9) and (10) the above equations implies

$$\varrho\frac{\partial u}{\partial t} = \eta_1\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right) + \frac{2\odot}{r}. \tag{18}$$

We consider the simple case in which the inital state is one of rest and the motion is set up by a uniform acceleration g of the outer boundary. We also suppose that motion takes place in the whole of the region $r_i < r < r_e$. The boundary conditions are then

$$\left.\begin{aligned} u &= 0, & t &= 0, & r_i &< r < r_e \\ u &= gt, & t &> 0, & r &= r_e \\ u &= 0, & t &> 0, & r &= r_i. \end{aligned}\right\} \tag{19}$$

The Laplace transform $\bar{u}(r, \xi) = \int_0^\infty u(r, t)\exp(-\xi t)dt$ of (18) aud (19) reduces the problem to the solution of the differential equation

$$\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r} - \left(\lambda^2 + \frac{1}{r^2}\right)\bar{u} = -\frac{2\odot}{\eta_1 \xi r},\ \lambda^2 = \frac{\varrho\xi}{\eta_1}, \tag{20}$$

with the boundary conditions $\bar{u} = 0$ on $r = r_i$ and $\bar{u} = g/\xi^2$ on $r = r_e$.

When this is done we have (Relton, 1946)

$$\bar{u} = c_1 I_1(\lambda r) + c_2 K_2(\lambda r) + \frac{2\odot}{\varrho\xi^2 r} \tag{21}$$

with I_1 and K_1 are modified Bessel functions of order one of the first and the second kinds respectively and the constants c_1 and c_2 are given by

$$Ec_1 = \{[(g/\xi^2) - (2\odot)/(\varrho\xi^2 r_e)]K_1(\lambda r_i)\} + \{(2\odot)/(\varrho\xi^2 r_i)]K_1(\lambda r_e)\} \tag{22}$$

$$-Ec_2 = \{[(g/\xi^2) - (2\odot)/(\varrho\xi^2 r_e)]I_1(\lambda r_i)\} + \{[(2\odot)/(\varrho\xi^2 r_i)]I_1(\lambda r_e)\} \tag{23}$$

with
$$E = I_1(\lambda r_e)K_1(\lambda r_i) - I_1(\lambda r_i)K_1(\lambda r_e). \tag{24}$$

The inverse Laplace transform of (21) gives the velocity distribution u. Since motion for only small values of time is of interest we expand various expressions for large values of ξ and then take the Inversion.

When asymptotic expansions (Relton, 1946)

$$I_1(x) = (2\pi x)^{-\frac{1}{2}}(1-3/(8x)+\ldots)\ \exp x,$$
$$K_1(x) = \pi^{\frac{1}{2}}(2x)^{-\frac{1}{2}}(1+3/(8x)+\ldots)\ \exp(-x),$$

and the formulae sinh x = cosh $x = \frac{1}{2}$ exp x for large x, are used, we get

$$K_1(\lambda r_i)I_1(\lambda r)-I_1(\lambda r_i)K_1(\lambda r) = (2\lambda)^{-1}(r_i r)^{-\frac{1}{2}}[1+\tfrac{3}{8}\lambda^{-1}(r_i^{-1}-r^{-1})]\ \exp[\lambda(r-r_i)] \quad (25)$$

$$E^{-1} = 2\lambda(r_i r_e)^{\frac{1}{2}}[1-\tfrac{3}{8}\lambda^{-1}(r_i^{-1}-r_e^{-1})]\ \exp[-\lambda(r_e-r_i)] \quad (26)$$

and similar results, neglecting terms of λ^{-2} and higher orders. With the help of results from (22) to (26), the relation (21) simplifies to

$$\begin{aligned}\bar{u} = \{&(r_e/r)^{\frac{1}{2}}\xi^{-2}[g-(2\odot)/(\varrho r_e)][1-\tfrac{3}{8}(\eta_1/\varrho)^{\frac{1}{2}}\xi^{-\frac{1}{2}}(r^{-1}-r_e^{-1})]\ \exp[-(r_e-r)\\ &(\varrho\xi/\eta_1)^{\frac{1}{2}}\}+\{[(2\odot)/(\varrho\xi^2)](r_i r)^{-\frac{1}{2}}[1-\tfrac{3}{8}(\eta_1/\varrho)^{\frac{1}{2}}\xi^{-\frac{1}{2}}(r_i^{-1}+r^{-1}-2r_e^{-1})]\\ &\exp[-(2r_e-r_i-r)(\varrho\xi/\eta_1)^{\frac{1}{2}}]\}+\{(2\odot)/(\varrho\xi^2 r)\}\end{aligned} \quad (27)$$

The inversion of the above relation gives (Crank, 1956)

$$\begin{aligned}(u/U) = {}&(ar_1)^{-\frac{1}{2}}(g_1-2Ba)\{4\tau\, i^2\,\mathrm{erfc}[\tfrac{1}{2}\tau^{-\frac{1}{2}}(a^{-1}-r_1)]\\ &-3(r_1^{-1}-a)\tau^{3/2}i^3\mathrm{erfc}[\tfrac{1}{2}\tau^{-\frac{1}{2}}(a^{-1}-r_1)\}]+2Br_1^{-\frac{1}{2}}\{4\tau i^2\,\mathrm{erfc}[\tfrac{1}{2}\tau^{-\frac{1}{2}}(2a^{-1}-r_1-1)]\\ &-3(1+r_1^{-1}-2a)\tau^{3/2}i^3\ \mathrm{erfc}\ [\tfrac{1}{2}\tau^{-\frac{1}{2}}(2a^{-1}-r_1-1)]\}+2B\tau r_1^{-1}\end{aligned} \quad (28)$$

with $$r = r_i r_1,\ t\eta = \varrho r_i^2\tau,\ g_1\eta_1 U = g\varrho r_i^2$$

while 'a' and 'B' are defined as before and U in this problem is be treated as a reference velocity used to define 'C'. In order to satisfy the condition of flow, viz. $e_{r\theta} > 0$ in the whole region $r_i < r < r_e$, we write

$$\begin{aligned}(u/U) = {}&4\tau(ar_1)^{-\frac{1}{2}}(g_1-2Ba)i^2\,\mathrm{erfc}[\tfrac{1}{2}\tau^{-\frac{1}{2}}(a^{-1}-r_1)]\\ &+8B\tau r_1^{-\frac{1}{2}}i^2\,\mathrm{erfc}[\tfrac{1}{2}\tau^{-\frac{1}{2}}(2a^{-1}-r_1-1)]+2B\tau r_1^{-1}\end{aligned} \quad (28a)$$

neglecting i^3 erfc, which means the same as retaining terms upto ξ^{-2} in (27). Noting the results (Crank, 1956)

$$d/dx\,(\mathrm{erfc}\,x) = -2\pi^{-\frac{1}{2}}\exp(-x^2),\ d/dx(i\mathrm{erfc}\,x) = -\mathrm{erfc}\,x,\ d/dx(i^2\,\mathrm{erfc}\,x) = -i\,\mathrm{erfc}\,x$$

we obtain the condition of flow ($e_{r\theta} \geq 0$) from (9) and (28a) as

$$\begin{aligned}(g_1-2Ba)a^{-\frac{1}{2}}\{&i\,\mathrm{erfc}[\tfrac{1}{2}\tau^{-\frac{1}{2}}i\,\mathrm{erfc}\,(a^{-1}-r_1)-3r_1^{-1}\tau^{\frac{1}{2}}i^2\,\mathrm{erfc}[\tfrac{1}{2}\tau^{-\frac{1}{2}}(a^{-1}-r_1)]\}\\ &+2B\{i\mathrm{erfc}[\tfrac{1}{2}\tau^{-\frac{1}{2}}(2a^{-1}-r_1-1)-3r_1^{-1}\tau^{\frac{1}{2}}i^2\,\mathrm{erfc}[\tfrac{1}{2}\tau^{-\frac{1}{2}}(2a^{-1}-r_1-1)]\\ &-\tau^{\frac{1}{2}}r_1^{-3/2}\}\geq 0, \qquad 1 \leq r_1 \leq a^{-1}.\end{aligned} \quad (29)$$

The positive terms in the left side of the above inequality decrease as r_1 decreases and hence to ensure the condition (29) we put the lowest value of r_1, *viz.* $r_1 = 1$ and obtain $\{(2Ba^{\frac{1}{2}})/g_1\} \leq F(a, \tau)$, where

$$\begin{aligned}F(a, \tau) = {}&\{i\,\mathrm{erfc}[\tfrac{1}{2}\tau^{-\frac{1}{2}}(a^{-1}-1)]-3\tau^{\frac{1}{2}}i^2\,\mathrm{erfc}[\tfrac{1}{2}\tau^{-\frac{1}{2}}(a^{-1}-1)]\}/\{a^{\frac{1}{2}}i\,\mathrm{erfc}[\tfrac{1}{2}\tau^{-\frac{1}{2}}(a^{-1}-1)]\\ &-3(a\tau)^{\frac{1}{2}}i^2\,\mathrm{erfc}[\tfrac{1}{2}\tau^{-\frac{1}{2}}(a^{-1}-1)]-i\,\mathrm{erfc}[\tau^{-\frac{1}{2}}(a^{-1}-1)]\\ &+8\tau^{\frac{1}{2}}i^2\,\mathrm{erfc}[\tau^{-\frac{1}{2}}(a^{-1}-1)]\pm\tau^{\frac{1}{2}}\}.\end{aligned} \quad (30)$$

To fix ideas if we take $a = 0{\cdot}5$ the condition reads as

$$(B/g_1) \leq 2^{-\frac{1}{2}}F(\tau) = \{F_1(\tau)/[F_1(\tau)-2^{\frac{1}{2}}F_2(\tau)]\} \tag{31}$$

with

$$F_1(\tau) = i\,\mathrm{erfc}\,(\tfrac{1}{2}\tau^{-\frac{1}{2}})-3\tau^{\frac{1}{2}}i^2\,\mathrm{erfc}(\tfrac{1}{2}\tau^{-\frac{1}{2}})$$

$$F_2(\tau) = i\,\mathrm{erfc}\,(\tau^{-\frac{1}{2}})-3\tau^{\frac{1}{2}}i^2\,\mathrm{erfc}\,(\tau^{-\frac{1}{2}})-\tau^{\frac{1}{2}}.$$

Table II gives the values of $F(\tau)/2^{\frac{1}{2}}$ for different values of τ. The negative values corresponding to $\tau \geq 1{\cdot}0$ are inadmissible, (B/g_1) being essentially positive. It is seen from the table that as τ increases, $F(\tau)/2^{\frac{1}{2}}$ gradually increases, takes a maximum value at a point in between $\tau = 0{\cdot}31$ and $\tau = 0{\cdot}39$ and then gradually decreases. This implies that for a prescribed value of $(B/g_1) = \{F(\tau)/2^{\frac{1}{2}}\}$, there are two values of τ between which $\{F(\tau)/2^{\frac{1}{2}}\} \geq (B/g_1)$ and hence the flow pattern is a possible one within this period only.

TABLE II

Values of $\{F(\tau)/2^{\frac{1}{2}}\}$.

τ	0.06	0.08	0.11	0.13	0.15	0.17
$F(\tau)/2^{\frac{1}{2}}$	0.0028	0.0053	0.0137	0.0175	0.0232	0 0290
τ	0.21	0.25	0.31	0.39	0.44	0.51
$F(\tau)/2^{\frac{1}{2}}$	0.0344	0.0394	0.0424	0.0421	0.0401	0.0362
τ	0.59	0.69	0.83		1.0	1.6
$F(\tau)/2^{\frac{1}{2}}$	0.0302	0.0214	0.0090		−0.0074	−0.1660

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THIN ANISOTROPIC ELASTIC SHELLS
I. CIRCULAR CYLINDER OF CYLINDRICAL AEOLOTROPY

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Abstract. The effect of anisotropy on the deformation of a thin circular cylindrical shell has been considered when it is bent by lateral loads only. The theory adopted is that given by Love. It is found that for a particular material the magnitude of deflection in anisotropy may be decreased from that in isotropy by 29% or even more, depending upon the polar angle. The less the polar angle the more the anisotropic effect.

1. Introduction. Beginning with the work of Aron (1874), Mathieu (1883), Love (1888) and Basset (1890) as re-examined by Krauss (1929), Trefftz (1935) Odquist (1937) and others, there exists a vast literature about the theory of stretching and bending of thin shells of isotropic material. Though an account of the general theory may be found in books of Love (1926), Flügge (1934), Biezeno and Grammel (1939) and Timoshenko (1940), special interest lies in the treatments of Reissner (1912), Meissner (1918), Schwerin (1917), Dischinger (1935), Havers (1935), Jakobsen (1937) as well as in those of Reutter (1942), Byrne (1944), Fadle (1944), Goldenweiser (1945), Rabotnove (1945) and Zerna (1949). The papers by Chien (1944) and Synge and Chien (1941) are remarkable for their complete generality. For fresh derivation and critical reviews, we may refer to papers of Truesdell (1945), Green and Zerna (1949) and Reissner (1941, 1946, 1948, 1950, 1955*a*, 1955*b*). Circular cylindrical shells which concern the present part of the paper have been dealt with by Teodorescu (1952), Grigolyuk (1953), Hoff and Kempner (1953), Wuest (1954), Ambartsumyan (1954) and Yuan and Ting (1957). In all cases, however, the material is assumed to be isotropic. In a general survey of orthotropic shells by Hildebrand, Reissner and Thomas (1949), more emphasis has been put on the effect of transverse shear stresses than on the effect of anisotropy. We shall investigate in a series of papers the modifications to be introduced when anisotropy of the material is taken into account. For the purpose of completeness, the equations of equilibrium, stress-strain relations etc. will be deduced from those of three-dimensional elasticity and usual assumptions will be introduced only at the particular stage where required. In this first part cirular cylinders with lateral loads will be discussed.

2. Equations of equilibrium. We consider a thin circular cylindrical shell of thickness $2h$, the radius of the middle surface in the unstrained state being 'a'. The position of a point on the middle surface may be specified by two coordinates x and φ, where the axis of x is taken parallel to the axis of the cylinder and φ is the angle between a pair of planes passing through the axis of the cylinder and the origin is taken on the middle surface. We take the axis of z along an inward drawn radius. The positive directions of the axes of x, φ and z are so chosen that they form a right-handed system. Thus instead of usual cylindrical coordinates $(r, \theta; z)$ we take (x, φ, z) where (r, θ, z) are replaced by $(a\text{-}z, \varphi, x)$ respectively. The linear element ds in this system between any two points is give by

$$ds^2 = dx^2+(a-z)d\varphi^2+dz^2. \tag{1}$$

The general equations of equilibrium for the stresses in curvilinear coordinates (Love, 1926) take the forms,

$$\left.\begin{aligned}
&\frac{\partial\sigma_x}{\partial x}+\frac{1}{a-z}\frac{\partial\tau_{x\varphi}}{\partial\varphi}+\frac{\partial\tau_{xz}}{\partial z}-\frac{1}{a-z}\tau_{xz}+F_x=0\\
&\frac{\partial\tau_{x\varphi}}{\partial x}+\frac{1}{a-z}\frac{\partial\sigma_\varphi}{\partial\varphi}+\frac{\partial\tau_{\varphi z}}{\partial z}-\frac{2}{a-z}\tau_{\varphi z}+F_\varphi=0\\
&\frac{\partial\tau_{zx}}{\partial x}+\frac{1}{a-z}\frac{\partial\tau_{\varphi z}}{\partial\varphi}+\frac{\partial\sigma_z}{\partial z}-\frac{1}{a-z}(\sigma_z-\sigma_\varphi)+F_z=0.
\end{aligned}\right\} \tag{2}$$

We multiply these equations by $(a-z)$ and intergrate with respect to z for the limits $(-h, h)$. We also multiply the first two equations by $z(a-z)$ and integrate for the same limits. The second operation is not performed on the third equation, for that will introduce quantities not required in the present theory. It is then obtained,

$$\left.\begin{aligned}
&\frac{\partial N_x}{\partial x}+\frac{1}{a}\frac{\partial N_{\varphi x}}{\partial\varphi}+p_x=0\\
&\frac{\partial N_{x\varphi}}{\partial x}+\frac{1}{a}\frac{\partial N_\varphi}{\partial\varphi}-\frac{1}{a}Q_\varphi+p_\varphi=0\\
&\frac{\partial Q_x}{\partial x}+\frac{1}{a}\frac{\partial Q_\varphi}{\partial\varphi}+\frac{1}{a}N_\varphi+p_z=0
\end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned}
&\frac{\partial M_x}{\partial x}+\frac{1}{a}\frac{\partial M_{\varphi x}}{\partial\varphi}-Q_x+m_x=0\\
&\frac{\partial M_{x\varphi}}{\partial z}-\frac{1}{a}\frac{\partial M_\varphi}{\partial\varphi}+Q_\varphi-m_\varphi=0
\end{aligned}\right\} \tag{4}$$

where the stress-resultants N_x, $N_{x\varphi}$, Q_x etc. and the stress couples M_x, $M_{x\varphi}$ etc. are defined as

$$\left.\begin{aligned}
N_x &= \int_{-h}^{h}\left(1-\frac{z}{a}\right)\sigma_x dz, & N_\varphi &= \int_{-h}^{h}\sigma_\varphi dz \\
N_{x\varphi} &= \int_{-h}^{h}\left(1-\frac{z}{a}\right)\tau_{x\varphi} dz, & N_{\varphi x} &= \int_{-h}^{h}\tau_{\varphi x} dz \\
Q_x &= \int_{-h}^{h}\left(1-\frac{z}{a}\right)\tau_{xz} dz, & Q_\varphi &= \int_{-h}^{h}\tau_{\varphi z} dz \\
M_x &= \int_{-h}^{h} z\left(1-\frac{z}{a}\right)\sigma_x dz, & M_\varphi &= \int_{-h}^{h} z\sigma_\varphi dz \\
M_{x\varphi} &= -\int_{h}^{h} z\left(1-\frac{z}{a}\right)\tau_{x\varphi} dz, & M_{\varphi x} &= \int_{-h}^{h} z\tau_{\varphi x} dz
\end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned}
p_x &= \left[\left(1-\frac{z}{a}\right)\tau_{xz}\right]_{-h}^{h} + \int_{-h}^{h}\left(1-\frac{z}{a}\right)F_x dz, \\
p_\varphi &= \left[\left(1-\frac{z}{a}\right)\tau_{\varphi z}\right]_{-h}^{h} + \int_{-h}^{h}\left(1-\frac{z}{a}\right)F_\varphi dz \\
p_z &= \left[\left(1-\frac{z}{a}\right)\tau_{zz}\right]_{-h}^{h} + \int_{-h}^{h}\left(1-\frac{z}{a}\right)F_z dz \\
m_x &= \left[z\left(1-\frac{z}{a}\right)\tau_{xz}\right]_{-h}^{h} + \int_{-h}^{h} z\left(1-\frac{z}{a}\right)F_x dz \\
m_\varphi &= \left[z\left(1-\frac{z}{a}\right)\tau_{\varphi z}\right]_{-h}^{h} + \int_{h}^{h} z\left(1-\frac{z}{a}\right)F_\varphi dz.
\end{aligned}\right\} \tag{6}$$

Elimination of Q_φ and Q_x between (3) and (4) gives

$$\begin{aligned}
&\frac{\partial N_x}{\partial x} + \frac{1}{a}\frac{\partial N_{\varphi x}}{\partial \varphi} + p_x = 0 \\
&\frac{\partial N_{x\varphi}}{\partial x} + \frac{1}{a}\frac{\partial N_\varphi}{\partial \varphi} + \frac{1}{a}\frac{\partial M_{x\varphi}}{\partial x} - \frac{1}{a^2}\frac{\partial M_\varphi}{\partial \varphi} + p_\varphi - \frac{1}{a}m_\varphi = 0 \\
&\frac{\partial^2 M_x}{\partial x^2} + \frac{1}{a}\frac{\partial^2 M_{x\varphi}}{\partial x\partial \varphi} - \frac{1}{a}\frac{\partial^2 M_{x\varphi}}{\partial x\partial \varphi} + \frac{1}{a^2}\frac{\partial^2 M_\varphi}{\partial \varphi^2} + \frac{1}{a}N_\varphi + p_z - \frac{\partial m_x}{\partial x} + \frac{1}{a}\frac{\partial m_\varphi}{\partial \varphi} = 0
\end{aligned} \tag{7}$$

These equations may be deduced from the general formulae given by Reissner (1941) and Hildebrand, Reissner and Thomas (1949). These are deduced in a different manner by Timoshenko (1940).

3 Strain-displacement relations. Using the general formulae in curvilinear coordinates (Love, 1926) the strain components are expressed in terms of displacement components as

$$\varepsilon_x = \frac{\partial u_x}{\partial x}, \quad \varepsilon_\varphi = \frac{1}{a-z}\left(\frac{\partial u_\varphi}{\partial \varphi} - u_z\right), \quad \varepsilon_z = \frac{\partial u_z}{\partial z}$$

$$\gamma_{\varphi z} = \frac{1}{a-z}\left(\frac{\partial u_z}{\partial \varphi} + u_\varphi\right) + \frac{\partial u_\varphi}{\partial z}, \quad \gamma_{zx} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \tag{8}$$

$$\gamma_{x\varphi} = \frac{\partial u_\varphi}{\partial x} + \frac{1}{a-z}\frac{\partial u_x}{\partial \varphi}.$$

4. Stress-strain relations. For cylindrically aeolotropic material the strain energy function W is given by

$$2W = c_{11}\varepsilon_x^2 + c_{22}\varepsilon_\varphi^2 + c_{33}\varepsilon_z^2 + 2c_{23}\varepsilon_\varphi\varepsilon_z + 2c_{31}\varepsilon_z\varepsilon_x + 2c_{12}\varepsilon_x\varepsilon_\varphi$$
$$+ c_{44}\gamma^2_{\varphi z} + c_{55}\gamma^2_{zx} + c_{66}\gamma^2_{x\varphi}, \qquad (c_{rs} = c_{sr})$$

The corresponding stress-strain relations are

$$\begin{aligned} \sigma_x &= c_{11}\varepsilon_x + c_{12}\varepsilon_\varphi + c_{13}\varepsilon_z & \tau_{\varphi z} &= c_{44}\gamma_{\varphi z} \\ \sigma_\varphi &= c_{21}\varepsilon_x + c_{22}\varepsilon_\varphi + c_{23}\varepsilon_z & \tau_{zx} &= c_{55}\gamma_{zx} \\ \sigma_z &= c_{31}\varepsilon_x + c_{32}\varepsilon_\varphi + c_{33}\varepsilon_z & \tau_{x\varphi} &= c_{66}\gamma_{x\varphi}. \end{aligned} \tag{9}$$

In a thin shell the normal stress σ_z is of smaller order of magnitude in comparison with normal sresses σ_x and σ_φ (Green and Zerna. 1950).

Hence it is customary to take in the stress-strain relations

$$\sigma_z = 0. \tag{10}$$

When the assumption (10) is used the elimination of ε_z reduces the first three relations of (9) to

$$\begin{aligned} \sigma_x &= s_{11}\varepsilon_x + s_{12}\varepsilon_\varphi \\ \sigma_\varphi &= s_{12}\varepsilon_x + s_{22}\varepsilon_\varphi \end{aligned} \tag{11}$$

where $\quad s_{11} = c_{11} - (c^2_{13}/c_{33}), \quad s_{22} = c_{22} - (c^2_{23}/c_{33}), \quad s_{12} = c_{12} - (c_{13}c_{23}/c_{33}).$

5. Usual assumptions. Let it be assumed that

$$\begin{aligned} u_x(x, \varphi, z) &= u(x, \varphi) + zu_1(x, \varphi) \\ u_\varphi(x, \varphi, z) &= v(x, \varphi) + zv_1(x, \varphi) \\ u_z(x, \varphi, z) &= w(x, \varphi), \end{aligned} \tag{12}$$

where (u, v, w) are the displacement components of the middle surface.

We also assume that the transverse shear stresses τ_{xz} and $\tau_{\varphi z}$ are zero, *i.e*,

$$\gamma_{xz} = \gamma_{\varphi z} = 0. \tag{13}$$

The assumptions (13) imply that straight lines initially normal to the unstrained middle surface remain straight after deformation and become normal to the strained middle surface (Bernoulli's law) (Tsuboi and Akino, 1955) while the third assumption of (12) put a severe restriction on the strain components ε_x and ε_φ viz, $c_{13}\varepsilon_x + c_{32}\varepsilon_\varphi = 0$ as evident from (8) and (9). This restriction is not retained in the classical theory of thin shell, thus introducing one of its drawbacks. Substituting (12) in the fourth and fifth relations of (8) and then using (13) we get

$$u_1 = -\frac{\partial w}{\partial x}, \quad v_1 = -\frac{1}{a}\left(\frac{\partial w}{\partial \varphi} + v\right). \tag{14}$$

The remaining relations of (8) with the help of (12) and (14) give

$$\varepsilon_x = \varepsilon_1 - z\chi_x, \qquad \varepsilon_\varphi = (\varepsilon_2 - z\chi_\varphi)\Big/\left(1 - \frac{z}{a}\right) \tag{15}$$

$$\gamma_{x\varphi} = \left[\varpi - \left(2 - \frac{z}{a}\right)\chi_{x\varphi}\right]\Big/\left(1 - \frac{z}{a}\right)$$

where

$$\varepsilon_1 = \frac{\partial u}{\partial x}, \quad \varepsilon_2 = \frac{1}{a}\left(\frac{\partial v}{\partial \varphi} - w\right), \quad \varpi = \frac{\partial v}{\partial x} + \frac{1}{a}\frac{\partial u}{\partial \varphi},$$

$$\chi_x = \frac{\partial^2 w}{\partial x^2}, \; \chi_\varphi = \frac{1}{a^2}\frac{\partial}{\partial \varphi}\left(\frac{\partial w}{\partial \varphi} + v\right), \; \chi_{x\varphi} = \frac{1}{a}\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial \varphi} + v\right). \tag{16}$$

The formulae (15) may be deduced from the general expressions given by Love (1926).

6. Love's first approximation. If the thickness of the shell is small in comparison with the radius, we may put in (5) and (15)

$$1 - \frac{z}{a} \simeq 1. \tag{17}$$

From (15) $$\varepsilon_x = \varepsilon_1 - z\chi_x, \; \varepsilon_\varphi = \varepsilon_2 - z\chi_\varphi, \; \gamma_{x\varphi} = \varpi - 2z\chi_{x\varphi}. \tag{18}$$

It is seen from (5), (9) and (13) that $Q_x = Q_\varphi = 0$ (19)

Even when (19) is not satisfied the relations (18) are assumed to be approximately correct. In that case Q_x and Q_φ are determined from (4). The other stress resultants and stress-couples are determined from (5), (11), (18) and the last relation of (9) as

$$\begin{aligned}
N_x &= 2h[s_{11}\varepsilon_1 + s_{12}\varepsilon_2], & N_\varphi &= 2h[s_{12}\varepsilon_1 + s_{22}\varepsilon_2] \\
N_{x\varphi} &= N_{\varphi x} = 2hc_{66}\varpi, \\
M_x &= -\tfrac{2}{3}h^3[s_{11}\chi_x + s_{12}\chi_\varphi], & M_\varphi &= -\tfrac{2}{3}h^3[s_{12}\chi_x + s_{22}\chi_\varphi] \\
M_{x\varphi} &= -M_{\varphi x} = \tfrac{4}{3}h^3c_{66}\chi_{x\varphi}.
\end{aligned} \tag{20}$$

The effective transverse shear forces V_x and V_φ and the effective tangential shear forces $V_{x\varphi}$ and $V_{\varphi x}$ defined as (Love, 1926),

$$V_x = Q_x - \frac{1}{a}\frac{\partial M_{x\varphi}}{\partial\varphi}, \quad V_\varphi = Q_\varphi + \frac{\partial M_{\varphi x}}{\partial x}$$

$$V_{x\varphi} = N_{x\varphi} + \frac{1}{a}M_{x\varphi}, \qquad V_{\varphi x} = -N_{\varphi x} + \frac{1}{a}M_{\varphi x} \tag{21}$$

are obtained from (4) and (20). Thus

$$V_x = -\tfrac{2}{3}h^3\left[s_{11}\frac{\partial\chi_x}{\partial x} + s_{12}\frac{\partial\chi_\varphi}{\partial x} + 4(c_{66}/a)\frac{\partial\chi_{x\varphi}}{\partial\varphi}\right] + m_x$$

$$V_\varphi = -\tfrac{2}{3}h^3\left[(s_{12}/a)\frac{\partial\chi_x}{\partial\varphi} + (s_{22}/a)\frac{\partial\chi_\varphi}{\partial\varphi} + 4c_{66}\frac{\partial\chi_{x\varphi}}{\partial x}\right] + m_\varphi \tag{22}$$

$$V_{x\varphi} = -V_{\varphi x} = 2hc_{66}[\varpi + \tfrac{2}{3}(h^2/a)\chi_{\varphi x}].$$

Substituting (16) in (20) and then (20) in (7) we get three differential equations satisfying the displacement components (u, v, w) of the middle surface as follows.

$$s_{11}\frac{\partial^2 u}{\partial x^2} + (c_{66}/a^2)\frac{\partial^2 u}{\partial\varphi^2} + a^{-1}(s_{12} + c_{66})\frac{\partial^2 v}{\partial x\partial\varphi} - a^{-1}s_{12}\frac{\partial w}{\partial x} + \tfrac{1}{2}(p_x/h) = 0$$

$$a^{-1}(s_{12} + c_{66})\frac{\partial^2 u}{\partial x\partial\varphi} + c_{66}\frac{\partial^2 v}{\partial x^2} + a^{-2}s_{22}\frac{\partial^2 v}{\partial\varphi^2} + \tfrac{1}{3}h^2a^{-2}(s_{12} + c_{66})\frac{\partial^3 w}{\partial x^2\partial\varphi} + \tfrac{1}{3}h^2a^{-4}s_{22}\frac{\partial^3 w}{\partial\varphi^3}$$

$$+ \tfrac{1}{2}h^{-1}(p_\varphi - a^{-1}m_\varphi) = 0$$

$$s_{12}\frac{\partial u}{\partial x} + (s_{22}/a)\frac{\partial v}{\partial\varphi} - \tfrac{1}{3}(h^2/a)\frac{\partial}{\partial\varphi}\left\{(s_{12} + 4c_{66})\frac{\partial^2 v}{\partial x^2} + (s_{22}/a^2)\frac{\partial^2 u}{\partial\varphi^2}\right\}$$

$$-(s_{22}/a)w - \tfrac{1}{3}ah^2\left\{s_{11}\frac{\partial^4 w}{\partial x^4} + 2a^{-2}(s_{12} + 2c_{66})\frac{\partial^4 w}{\partial x^2\partial\varphi^2} + (s_{22}/a^4)\frac{\partial^4 u}{\partial\varphi^4}\right\}$$

$$+ \tfrac{1}{2}(a/h)\left(p_z - \frac{\partial m_x}{\partial x} + a^{-1}\frac{\partial m_\varphi}{\partial\varphi}\right) = 0. \tag{23}$$

In writing the second equation of (23) the assumption (17) has been used. The results for isotropic material will be obtained by setting $c_{11} = c_{22} = c_{33} = \lambda + 2\mu$, $c_{44} = c_{55} = c_{66} = \mu$, $c_{23} = c_{31} = c_{12} = \lambda$ where (λ, μ) are Lame constants.

7. Simply supported shell with sinusoidal loads. Let the shell of which the edges are given by $x = 0$, $x = l$, $\varphi = 0$ and $\varphi = \alpha$ be bent by a normal load only so that $p_x = p_\varphi = m_x = m_\varphi = 0$ and let

$$p_z = p_0 \sin(\pi\varphi/\alpha)\sin(\pi x/l). \tag{24}$$

If the shell be simply supported the boundary conditions are (Love, 1926)

$$w = M_x = N_x = 0 \quad \text{on} \quad x = 0,\ x = l \tag{25}$$

$$w = M_\varphi = N_\varphi = 0 \quad \text{on} \quad \varphi = 0,\ \varphi = \alpha \tag{26}$$

$$V_{x\varphi} = 0 \qquad \text{on} \quad x = 0,\ x = l \tag{27}$$

$$V_{\varphi x} = 0 \qquad \text{on} \quad \varphi = 0,\ \varphi = \alpha. \tag{28}$$

The displacement components

$$\begin{aligned} u &= A \sin(\pi\varphi/\alpha)\cos(\pi x/l) \\ v &= B \cos(\pi\varphi/\alpha)\sin(\pi x/l) \\ w &= C \sin(\pi\varphi/\alpha)\sin(\pi x/l) \end{aligned} \tag{29}$$

where A, B and C are constants may be easily verified to satisfy the conditions (25) and (26). Substitution of (29) in (23) yields

$$\begin{aligned} &\{(s_{11}/l^2) + [c_{66}/(a^2\alpha^2)]\}A + \{(s_{12}+c_{66})/(a\alpha l)\}B + [s_{12}/(\pi a l)]C = 0 \\ &\{s_{12}+c_{66})/(a\alpha l)\}A + \{[s_{22}/(a^2\alpha^2)] + (c_{66}/l^2)\}B + \tfrac{1}{3}\{(\pi h^2)/(a^2\alpha)\}\{[(s_{12}+2c_{66})/l^2] + [s_{22}/(a^2\alpha^2)]\}C = 0 \\ &(s_{12}/l)A + \{s_{22} + \tfrac{1}{3}\pi^2h^2[l^{-2}(s_{12}+4c_{66}) + a^{-2}\alpha^{-2}s_{22}]\}\{B/(a\alpha)\} + \{[s_{22}/(\pi a)] + \tfrac{1}{3}ah^2\pi^3[(s_{11}/l^4) \\ &\qquad + 2(a\alpha l)^{-2}(s_{12}+2c_{66}) + (a\alpha)^{-4}s_{22}]\}C - \tfrac{1}{2}\{(ap_0)/(\pi h)\} = 0. \end{aligned} \tag{30}$$

Equations in (30) determine the constants A, B and C. It may be pointed out that the state given by (29) and (30) does not satisfy the boundary conditions (27) and (28) exactly. However the average values of $V_{x\varphi}$ and $V_{\varphi x}$ on the corresponding edges as determined by the integrals $\int_0^\alpha V_{x\varphi}\,d\varphi$ and $\int_0^l V_{\varphi x}\,dx$ vanish, thus showing that each of these tangential shear forces forms a self-equilibrating system on the seperate edges of the shell. Hence by Saint Venant's principle we may assume that the boundary conditions (27) and (28) are satisfied so long as the edge-effects are not considered. Using (29) and (16) in (20) and the first two equations of (22) it is obtained that

$$\begin{aligned} M_x &= \tfrac{2}{3}\pi^2h^3\{(\pi a^2\alpha)^{-1}s_{12}B + [s_{11}/l^2) + s_{12}(a\alpha)^{-2}]C\}\sin(\pi\varphi/\alpha)\sin(\pi x/l) \\ M_\varphi &= \tfrac{2}{3}\pi^2h^3\{(\pi a^2\alpha)^{-1}s_{22}B + [(s_{12}/l^2) + s_{22}(a\alpha)^{-2}]C\}\sin(\pi\varphi/\alpha)\sin(\pi x/l) \\ M_{x\varphi} &= -M_{\varphi x} = \tfrac{4}{3}\pi^2h^3c_{66}(a\alpha l)^{-1}\{(\alpha/\pi)B + C\}\cos(\pi\varphi/l)\cos(\pi x/l) \\ V_x &= \tfrac{2}{3}\pi^3h^3\{(\pi a^2\alpha l)^{-1}(s_{12}+4c_{66})B + [(s_{11}/l^3) + (a^2\alpha^2 l)^{-1}(s_{12}+4c_{66})]C\}\sin(\pi\varphi/\alpha)\cos(\pi x/l) \\ V_\varphi &= \tfrac{2}{3}\pi^3h^3\{(\pi a)^{-1}[(a\alpha)^{-2}s_{22} + 4l^{-2}c_{66}]B + [(a\alpha)^{-3}s_{22} + (l^2a\alpha)^{-1}(s_{12}+4c_{66})]C\}\cos(\pi\varphi/\alpha)\sin(\pi x/l). \end{aligned} \tag{31}$$

In the particular case when $l = a\alpha$, the equations in (30) simplify to

$$\begin{aligned} &(1+\lambda_{11})A + (1+\lambda_{12})B + \lambda_{12}C[l/(\pi a)] = 0 \\ &(1+\lambda_{12})A + (1+\lambda_{22})B + \tfrac{1}{3}(h^2/a^2)(2+\lambda_{12}+\lambda_{22})C[(\pi a)/l] = 0 \\ &\lambda_{12}A + \{\lambda_{22} + \tfrac{1}{3}\pi^2(h^2/l^2)(4+\lambda_{12}+\lambda_{22})\}B + \{[l/(\pi a)]\lambda_{22} + \tfrac{1}{3}\pi^3ah^2l^{-3}(4+\lambda_{11}+3\lambda_{12})\}C \\ &\qquad - \{(alp_0)/(2\pi h c_{66})\} = 0 \end{aligned} \tag{32}$$

where

$$\lambda_{11} = s_{11}/c_{66}, \quad \lambda_{22} = s_{22}/c_{66}, \quad \lambda_{12} = s_{12}/c_{66}. \tag{33}$$

In the isotropic material

$$\lambda_{11} = \lambda_{22} = 4(c_{12}+c_{66})/(c_{12}+2c_{22}),\ \lambda_{12} = 2c_{66}/(c_{12}+2c_{66}). \tag{34}$$

8. Numerical results. For barytes (Love, 1926) we have

$$c_{11} = 907,\ c_{22} = 800,\ c_{33} = 1074,\ c_{23} = 273,\ c_{31} = 275$$

$$c_{12} = 468,\ c_{44} = 122,\ c_{55} = 293,\ c_{66} = 283, \tag{35}$$

where the constants are expressed in terms of an unit stress of grammes' weight per square centimetre. These give

$$\lambda_{11} = 2{\cdot}9561,\ \lambda_{22} = 2{\cdot}8268,\ \lambda_{12} = 1{\cdot}6537. \tag{36}$$

In the isotropic case, calculating $\lambda_{11} = \lambda_{22}$ and λ_{12} from (34) and (35) we get

$$\lambda_{11} = \lambda_{22} = 2{\cdot}9052,\ \lambda_{12} = 0{\cdot}5474. \tag{36a}$$

Since it is generally conceded that the shell is no longer thin when $(h/a) > 1/30$, we take an extreme value, viz. $(h/a) = 0{\cdot}03$. The equations (32) now give A, B and C as

$$(A_a,\ B_a,\ C_a) = (\bar{u}_a,\ \bar{v}_a,\ \bar{w}_a)\tfrac{1}{2}(hc_{66})^{-1}a^2p_0 \tag{37}$$

$$(A_i,\ B_i,\ C_i) = (\bar{u}_i,\ \bar{v}_i,\ \bar{w}_i)\tfrac{1}{2}(hc_{66})^{-1}a^2p_0 \tag{37a}$$

where A_i and A_a denote the values of A for the isotropic and anisotropic cases respectively with similar notations for other quantities and $\bar{u}_a$, $\bar{u}_i$...are the amplitude constants depending upon α. The following table gives the values of $\bar{u}_a, \bar{u}_i$...as functions of $(\alpha/\pi) = l/(\pi a)$. It is seen that the absolute value of $\bar{u}_a$ and $\bar{v}_a$ are always greater than $\bar{u}_i$ and $\bar{v}_i$ resphctively, while $\bar{w}_a$ is less then $\bar{w}_i$. For lateral loads, as in the present illustration, normal deflection w are of primary consideration and the anisotropic effect on it may be as far as to decrease the value by 29% corresponding to the polar angle $\alpha = 0{\cdot}05\pi$.

TABLE

Values of amplitude constants

$\frac{\alpha}{\pi} = \frac{l}{\pi a}$		0.05	0.08	0.10	0.20	0.50	0.70
	$\bar{u}_a$	0.0	−0.0006	−0.0018	−0.0301	−0.1248	−0.1773
	$\bar{u}_i$	0.0	−0.0002	−0.0005	−0.0070	−0.0358	−0.0385
	$\bar{v}_a$	0.0	0.0003	0.0016	0.0196	0.0862	0.1227
	$\bar{v}_i$	0.0	0.0	0.0001	0.0023	0.0105	0.0151
	$\bar{w}_a$	0.0017	0.0111	0.0258	0.1888	0.3208	0.3247
	$\bar{w}_i$	0.0024	0.0153	0.0351	0.2184	0.3287	0.3318
Decrease in	$\bar{w}_i$	29%	28%	26%	14%	3%	2%

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NOTE ON A FORMULA OF RAINVILLE

BY

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(Communicated by the Secretary—Received January 28, 1959)

Rainville (1945) has proved the formula

$$\left(\frac{\sin\beta}{\sin\alpha}\right)^n P_n(\cos\alpha) = \sum_{r=0}^{n} \binom{n}{r} \left(\frac{\sin(\beta-\alpha)}{\sin\alpha}\right)^r P_{n-r}(\cos\beta \tag{1}$$

In a recent paper, Bhonsle (1957) has made some applications of this formula

We should like to point out that (1) can be generalized to hold for the ultraspherical polynomials $C_n^\lambda(x)$ defined by

$$(1-2xt+t^2)^{-\lambda} = \sum_{n=0}^{\infty} t^n C_n^\lambda(x).$$

Indeed, since [Erdelyi *et al*, 1955, p. 262, formula (10)]

$$\frac{e^{t\cos\beta}}{(\frac{1}{2}t^2\sin^2\beta)} J_\lambda(t\sin\beta) = \sum_{n=0}^{\infty} \frac{t^n C_n^{\lambda-\frac{1}{2}}(\cos\beta)}{(2\lambda+1)_n},$$

we have
$$\sum_{r=0}^{\infty} \frac{t^r}{r!}\left(\frac{\sin(\beta-\alpha)}{\sin\alpha}\right)^r \sum_{n=0}^{\infty} \frac{t^n C_n^{\lambda+\frac{1}{2}}(\cos\beta)}{(2\lambda-1)_n} \; \frac{e^{t\sin\beta\cos\alpha/\sin\alpha}}{(\frac{1}{2}t^2\sin^2\beta)^\lambda} J_\lambda(t\sin\beta)$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{(2\lambda-1)^n} \left(\frac{\sin\beta}{\sin\sigma}\right)^n C_n^{\lambda-\frac{1}{2}}(\cos\alpha).$$

Consequently, on equating coefficients, we get

$$\left(\frac{\sin\beta}{\sin\alpha}\right)^n C_n^{\lambda+\frac{1}{2}}(\cos\alpha) = \sum_{r=0}^{n} \binom{2\lambda-n}{r} \left(\frac{\sin(\beta-\alpha)}{\sin\alpha}\right)^r C_{n-r}^{\lambda+\frac{1}{2}}(\cos\beta). \tag{2}$$

For $\lambda = 0$, (2) evidently reduces to (1). If we recall that (Szegö, 1939, pp. 80, 81

$$C_n^1(\cos\alpha) = \frac{\sin(n+1)\alpha}{\sin\alpha},$$

$$\lim_{y=0} \lambda^{-1} C_n(\cos\alpha) = 2/n \cos n\alpha \qquad (n \geq 1)$$

we get the special cases

$$\left(\frac{\sin\beta}{\sin\alpha}\right)^n \sin n\alpha = \sum_{r=0}^{n-1} \binom{n}{r} \left(\frac{\sin(\beta-\alpha)}{\sin\alpha}\right)^r \sin(n-r)\beta \tag{3}$$

$$\left(\frac{\sin\beta}{\sin\alpha}\right)^n \cos n\alpha = \sum_{r=0}^{n} \binom{n}{r} \left(\frac{\sin(\beta-\alpha)}{\sin\alpha}\right)^r \cos(n-r)\beta \tag{4}$$

Indeed (3) and (4) are contained in the obvious identity

$$\left(\frac{\sin\beta}{\sin\alpha}\right)^n e^{ni\alpha} = \sum_{r=0}^{n} \binom{n}{r} \left(\frac{\sin(\beta-\alpha)}{\sin\alpha}\right)^r e^{(n-r)i\beta}$$

$$= \left(\frac{\sin(\beta-\alpha)}{\sin\alpha} + e^{i\beta}\right)^n.$$

We remark that for $\beta = 2\alpha$, (2) reduces to

$$(2\cos\alpha)^n . C_n^{\lambda+\frac{1}{2}}(\cos\alpha) = \sum_{r=0}^{n} \binom{2\lambda+n}{r} C_{n-r}^{\lambda+\frac{1}{2}}(\cos 2\alpha). \tag{5}$$

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SOME RESULTS ON A CLASS OF NON-ABSOLUTELY CONTINUOUS FUNCTIONS

By

P. L. Ganguli, *Calcutta*

(Received—April, 18, 1959)

We discussed in a previous article (Sengupta and Ganguli, 1959) how starting with a series $\Sigma a_n = 1$ with $a_n > 0$ and $a_n > R_n = a_{n+1} + a_{n+2} + \ldots\ldots$, we may construct a class of functions $f(x)$ each steadily increasing and continuous in $0 \leq x \leq 1$ but not absolutely continuous in the interval.

In this note we propose to study the behaviour of the inverse functions of the above class of functions. We shall, in particular, show that the inverse functions are absolutely continuous in $0 \leq x \leq 1$.

Secondly, the bracketed series $(a_1 - R_1) + 2(a_2 - R_2) + 2^2(a_3 - R_3) + \ldots$ which emerged in the course of our discussion was proved to be convergent having a sum less than unity. In fact, the sum is $\int_0^1 f'(x)dx < 1$ and since $f(1) - f(0) = 1$, it follows that $f(x)$ is not an integral. We shall consider the behaviour in regard to convergence of the above series when the brackets are removed and also certain other allied problems in this connection.

We also discuss in this note how we may construct a series $\Sigma a_n = 1$ with $a_n > 0$, $a_n > R_n$ such that the series will enable us to define a class, having the power of the continuum, of steadily increasing continuous functions in $0 \leq x \leq 1$ and whose inverses are also absolutely continuous. We shall discuss the nature of convergence of the above series in this case also.

Lastly, we have discussed how we can construct, out of two infinite series of the above type, a more general class of functions which are steadily increasing and continuous but not absolutely so and of which the class considered previously form a particular case.

I

The key to the construction of the class of functions, referred to in the first paragraph of this note, was as follows.

We take an arbitrary convergent series of positive terms $\Sigma b_n = c < 1$ and construct another series of positive terms Σa_n such that

(*i*) $\Sigma a_n = 1$

(*ii*) $a_n > R_n = a_{n+1} + a_{n+2} + \ldots$

(*iii*) $0 < a_n - R_n < b_n/2^{n-1}$

Now, if S denotes the Cantor set on $0 \leq x \leq 1$ and $\alpha \in S$, then α can always be expressed uniquely in the form

$$\alpha = \cdot d_1 d_2 d_3 \ldots \qquad \text{(scale of 3)}$$

where each $d = 0$ or 2

We then define a function $f(x)$ on $0 \leq x \leq 1$ as follows·

If $x \in S$ and if $x = \cdot d_1 d_2 d_3 \ldots$ (scale of 3)

where $d_n = 0$ or 2 we take

$$\begin{aligned} f(x) = \Sigma a_n' \text{ where } a_n' &= 0 \text{ if } d_n = 0 \\ &= a_n \text{ if } d_n = 2 \end{aligned}$$

This defines $f(x)$ on S

Next, we define $f(x)$ on the complementary set CS according to the following scheme.

Let $\delta_1, \delta_2, \delta_3, \ldots$ be the everywhere dense set of enumerable open intervals constituting CS. Any interval δ_k is such that its end points ξ and η $(\xi < \eta)$ belong to S and such that $f(\xi) < f(\eta)$.

In the interior of δ_k we take

$$f(x) = f(\xi) + (x - \xi)\frac{f(\eta) - f(\xi)}{\eta - \xi}$$

It has been shown that the function $f(x)$ so defined is steadily increasing and continuous on $0 \leq x \leq 1$ but not absolutely continuous. Also we have shown that on each of the 2^{n-1} intervals of the nth order

$$f'(x) = 3^n(a_n - R_n)$$

Let us now consider the inverse of the above function $y = f(x)$, say, $x = \varphi(y)$ on $0 \leq y \leq 1$. The inverse function is obviously continuous and steadily increasing in $0 \leq y \leq 1$.

It is easy to show that each of the 2^{n-1} intervals of the nth order is mapped into an interval of length $(a_n - R_n)$. Also over each of these intervals

$$\varphi'(y) = \frac{1}{f'(x)} = \frac{1}{3^n(a_n - R_n)}$$

Hence
$$\int_0^1 \varphi'(y)dy \geq 1$$

But
$$\int_0^1 \varphi'(y)dy \leq \varphi(1) - \varphi(0) = 1$$

Therefore
$$\int_0^1 \varphi'(y)dy = 1 = \varphi(1) - \varphi(0)$$

It follows that the inverse function is absolutely continuous though the original function $f(x)$ is not so.

The conclusion that we naturally draw from the above fact is that there are some sets of measure zero on $0 \leq x \leq 1$ whose maps by the function $f(x)$ are not sets of measure zero while the map of every set of measure zero on $0 \leq y \leq 1$ by the inverse function $\varphi(y)$ is a set of measure zero.

Let us examine in particular the set $f(S)$ where S is the Cantor set on $0 \leq x \leq 1$. We know that $mS = 0$ and since the set is closed and $f(x)$ is continuons, the map $f(S)$ of S is also a closed set and hence measurable.

Now, $$mf(S) = 1 - [(a_1 - R_1) + 2(a_2 - R_2) + 2^2(a_3 - R_3) + \dots]$$

But $$0 < a_n - R_n < b_n/2^{n-1}$$

Hence $$mf(S) > 1 - (b_1 + b_2 + b_3 + \dots) = 1 - c > 0$$

Thus the set S of measure zero is mapped by $f(x)$ on a set of positive measure which incidentally establishes the fact that $f(x)$ is not absolutely continuous. This may be taken to be an alternative proof of the result obtained by Sen Gupta and Ganguli (1959)

The above results can be expressed in the following form.

If $$\Sigma a_n = 1,\ a_n > 0,\ a_n \geq R_n = a_{n+1} + a_{n+2} + \dots$$
$$0 < a_n - R_n < b_n/2^{n-1} \text{ where } \Sigma b_n = c < 1,$$

then the set E of numbers $\Sigma a_n'$ where $a_n' = 0$ or a_n at pleasure, is a set of positive measure.

Now, Prof. K. Menon (1948) has shown that
$$mE = \lim_{n \to \infty} 2^n R_n$$

Again, we have $$0 < a_n - R_n < b_n/2^{n-1}$$

Therefore $$\lim 2^n a_n = \lim 2^n R_n = mE > 0$$

As a corollary it immediately follows that, for any series $\Sigma a_n = 1$ with $a_n > 0$, $a_n > R_n = a_{n+1} + a_{n+2} + \dots$ and $0 < a_n - R_n < b_n/2^{n-1}$, $\lim 2^n a_n$ is positive and equals the measure of the set $f(S)$.

II

Let us consider the series of positive terms
$$(a_1 - R_1) + 2(a_2 - R_2) + 2^2(a_3 - R_3) + \dots \tag{A}$$
which is obviously convergent having the sum $1 - mf(S)$. As $\lim 2^{n-1} R_n$ is not zero, the series obtained by removing the brackets is not convergent.

If a new series is obtained by bracketing the terms in a different way, then the question naturally arises whether the new series converges and if convergent what will be its sum.

Let us, in particular, examine the following series
$$a_1 + (2a_2 - R_1) + (2^2 a_3 - 2R_2) + \dots + (2^{n-1} a_n - 2^{n-2} R_{n-1}) + \dots \tag{B}$$

If S_n and T_n denote the nth partial sums of the series (A) and (B) respectively then

$$T_n = S_n + 2^{n-1} R_n \to 1 - \tfrac{1}{2} m f(S)$$

Thus the series (A) and (B) have different sums.

Again, we have the following double series corresponding to the series (A):

$$\begin{array}{l} a_1 - a_2 - a_3 - a_4 - a_5 - \dots \\ 0 + 2a_2 - 2a_3 - 2a_4 - 2a_5 - \dots \\ 0 + 0 + 2^2 a_3 - 2^2 a_4 - 2^2 a_5 - \dots \\ 0 + 0 + 0 + 2^3 a_4 - 2^3 a_5 - \dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \end{array}$$

The sum of this double series, by columns, is $\Sigma a_n = 1$, while its sum by rows is obviously $1 - mf(S)$.

Again, the sums of the successive rows of the above double series, when each term is taken positive, are respectively

$$1,\ 2R_1,\ 2^2 R_2,\ 2^3 R_3, \dots$$

The series $1 + 2R_1 + 2^2 R_2 + 2^3 R_3 + \dots$ is clearly not convergent. It follows that the conditions for Cauchy's double series Theorem do not hold. The curious behaviour of the series (A) is not, therefore, unexpected.

III

We shall now construct a series $\Sigma a_n = 1$, $a_n > 0$ such that the corresponding function $f(x)$ defined on $0 \leq x \leq 1$ according to the above plan will be steadily increasing and absolutely continuous on $0 \leq x \leq 1$ and such that the inverse function will also have the same property. For this purpose we shall have to construct the series Σa_n in such a manner that the non-dense perfect set $\Sigma a_n'$, where $a_n' = 0$ or a_n, is of measure zero. Prof. K. Menon (1948) also discusses how this can be done and his method is remarkably simple. We, however, follow a different method.

We start with an arbitrary sequence of steadily decreasing positive numbers

$$b_1, b_2, b_3, \dots \ (b_1 < 1)$$

tending to zero as limit. Then we choose a sequence of positive numbers (a_n) successively according to the following scheme.

$$\max(\tfrac{1}{2},\ 1 - b_1/2) < a_1 < 1$$

$$\max\left(\frac{1-a_1}{2},\ 1 - a_1 - \frac{b_2}{2^2}\right) < a_2 < 1 - a_1$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$\max\left(\frac{1 - a_1 - a_2 - \dots - a_{n-1}}{2},\ 1 - a_1 - a_2 - \dots - a_{n-1} - \frac{b_n}{2^n}\right)$$

$$< a_n < 1 - a_1 - a_2 - \dots - a_{n-1}$$

and so on.

It is evident that each of the elements of the sequence $\{a_n\}$ can be chosen in a number of ways having the power of the continuum.

It is easy to verify that

$$\Sigma a_n = 1,\ a_n > R_n \text{ and } \lim 2^n R_n = 0$$

and it follows that the measure of the set $f(S)$ is zero.

Now, let us take any set F of measure zero on $0 \leq x \leq 1$. We can write

$$F = F.S + F.(CS)$$

Each of the sets $F.S$ and $F.CS$ is obviously of measure zero and clearly

$$f(F) = f(F.S) + f(F.CS)$$

Obviously, $m\{f(F.S)\} = 0$ and it is easily verified that $mf(F.CS) = 0$.

It follows that $mf(F) = 0$. This shows that any set of measure zero on $0 \leq x \leq 1$ is mapped by $f(x)$ into a set of measure zero. Hence the function $f(x)$, as constructed above, is absolutely continuous on $0 \leq x \leq 1$ and it is easy to show (as in Sec I) that the inverse function is also absolutely continuous.

Since $f(x)$ is absolutely continuous on $0 \leq x \leq 1$,

$$\int_0^1 f'(x)dx = (a_1 - R_1) + 2(a_2 - R_2) + \ldots + 2^{n-1}(a_n - R_n) + \ldots$$
$$= f(1) - f(0) = 1$$

As a corollary, it follows that

$$\lim 2^n a_n = \lim 2^n R_n = 0.$$

In this case, the series $\Sigma 2^{n-1}(a_n - R_n)$ converges even though the brackets be removed and the sum equals 1.

IV

In this section we propose to indicate how we may construct, out of two infinite series, a more general class of functions which are steadily increasing and continuous but not absolutely continuous. This class will include, as a particular case, the class of functions discussed by Sen Gupta and Ganguli (1959).

We construct two series Σa_n and ΣA_n of positive terms such that $\Sigma a_n = 1$, $\Sigma A_n = 1$ with

$$a_n > r_n = a_{n+1} + a_{n+2} + \ldots$$
$$A_n > R_n = A_{n+1} + A_{n+2} + \ldots$$

Starting with a steadily decreasing sequence of positive numbers b_1, b_2, b_3 tending to zero we choose $a_1, a_2, a_3, \ldots$ successively such that

$$\max\left(\tfrac{1}{2},\ 1 - b_1/2\right) < a_1 < 1$$

$$\max\left(\frac{1 - a_1 - a_2 - \ldots - a_{n-1}}{2},\ 1 - a_1 - a_2 - \ldots - a_{n-1} - b_n/2^n\right)$$
$$< a_n < 1 - a_1 - a_2 - \ldots - a_{n-1} \qquad (n = 2, 3, \ldots)$$

This ensures that

$$\Sigma a_n = 1,\ a_n > r_n \text{ and } \lim 2^n r_n = 0$$

Again, starting with a series Σc_n of positive terms such $\Sigma c_n = c < 1$, we choose $A_1, A_2, A_3, \ldots$ successively satisfying the conditions

$$\tfrac{1}{2} < A_1 < \frac{1+c_1}{2}$$

$$\frac{1-A_1-A_2-\ldots-A_{n-1}}{2} < A_n \leq \min\left\{\frac{1-A_1-A_2-\ldots-A_{n-1}}{2} + \frac{c_n}{2^n},\ 1-A_1-A_2-\ldots-A_{n-1}\right\}$$

This ensures that

$$\Sigma A_n = 1,\ A_n > R_n \text{ and } 0 < A_n - R_n < c_n/2^{n-1}$$

Let us now denote by S the set $\{\Sigma a_n'\}$ where

$$a_n' = a_n \text{ or } 0 \quad \text{at pleasure}$$

and by T the set $\{\Sigma A_n'\}$ where

$$A_n' = A_n \text{ or } 0 \quad \text{at pleasure.}$$

By what we have already proved

$$m(S) = 0 \text{ and } m(T) \geq 0$$

We now define a function $f(x)$ on $0 \leq x \leq 1$ which sends the set S onto the set T as follows:

If $\quad x = u_1 + u_2 + u_3 + \ldots,$ where $u_n = a_n$ or 0

we set $\quad f(x) = v_1 + v_2 + v_3 + \ldots$

where $\quad v_n = A_n$ if $u_n = a_n$

$\quad = 0$ if $u_n = 0$

By this mapping r_i and a_i are mapped respectively into R_i and A_i.

The set S splits the interval $0 \leq x \leq 1$ into complementary everywhere dense open non-overlapping intervals I of the type

$$u_1 + u_2 + \ldots + u_k + r_{k+1} < x < u_1 + u_2 + \ldots + u_k + a_{k+1}$$

where $\quad u_i = 0$ or a_i, $\quad i = 1, 2, \ldots, k$

The interval J of CT given by

$$v_1 + v_2 + \ldots + v_k + R_{k+1} < y < v_1 + v_2 + \ldots + v_k + A_{k+1}$$

where $\quad v_i = A_i$ if $u_i = a_i$

$\quad = 0$ if $u_i = 0, \quad i = 1, 2, \ldots, k$

will be said to correspond to the interval I of CS.

Any point of I is obviously of the form

$$x = u_1 + u_2 + \ldots + u_k + r_{k+1} + \theta(a_{k+1} - r_{k+1}),\ 0 \leq \theta < 1.$$

We define $f(x)$ on I as follows:

If $$x = u_1 + u_2 + \dots + u_k + r_{k+1} + \theta(a_{k+1} - r_{k+1})$$

we take $$f(x) = v_1 + v_2 + \dots + v_k + R_{k+1} + \theta(A_{k+1} - R_{k+1})$$

where $$v_i = A_i, \text{ if } u_i = a_i$$ $$= 0, \text{ if } u_i = 0$$

The function $f(x)$ thus defined maps the intervals of CS into corresponding intervals of CT. Clearly $f(x)$ is steadily increasing and continuous on $0 \leq x \leq 1$ with $f(0) = 0$, $f(1) = 1$ and since $m(S) = 0$ but $m(T) > 0$, it follows that $f(x)$ is not absolutely continuous.

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ON COMPRESSIONAL WAVES IN THE DIRECTION OF THE AXIS OF A CYLINDRICAL HOLE IN AN ANISOTROPIC MEDIUM HAVING HEXAGONAL SYMMETRY

By

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Introduction. Biot (1952) investigated the possibility and nature of compressional waves in an isotropic elastic medium having a circular cylindrical hole in it. The existence of such waves when the hole is situated in an anisotropic medium with hexagonal symmetry has been examined in this paper.

Formulation of the Problem. In cylindrical coordinates r, θ, z, the strain-energy function W is

$$W = \tfrac{1}{2}c_{11}(e_{rr}^2 + e_{\theta\theta}^2) + \tfrac{1}{2}c_{33}\,e_{zz}^2 + c_{13}(e_{rr}+e_{\theta\theta})e_{zz} + c_{12}e_{rr}e_{\theta\theta} + \tfrac{1}{2}c_{44}e_{rz}^2 + \tfrac{1}{4}(c_{11}-c_{12})e_{r\theta}^2 \tag{1}$$

The stress equations of motion in cylindrical coordinates in the absence of body forces are

$$\left.\begin{aligned}
&\frac{\partial\widehat{rr}}{\partial r} + \frac{1}{r}\frac{\partial\widehat{r\theta}}{\partial\theta} + \frac{\partial\widehat{rz}}{\partial z} + \frac{\widehat{rr}-\widehat{\theta\theta}}{r} = \varrho\frac{\partial^2 u_r}{\partial t^2}\\
&\frac{\partial\widehat{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\widehat{\theta\theta}}{\partial\theta} + \frac{\partial\widehat{\theta z}}{\partial z} + \frac{2\widehat{r\theta}}{r} = \varrho\frac{\partial^2 u_\theta}{\partial t^2}\\
&\frac{\partial\widehat{rz}}{\partial r} + \frac{1}{r}\frac{\partial\widehat{\theta z}}{\partial z} + \frac{\partial\widehat{zz}}{\partial z} + \frac{\widehat{rz}}{r} = \varrho\frac{\partial^2 u_z}{\partial t^2}
\end{aligned}\right\} \tag{2}$$

u_r, u_θ, u_z being the displacement components.

We shall consider a compressional wave propagating in the direction of the axis of cylindrical hole $r = a$. We therefore seek solutions of equation (2) of the type

$$[u_r, u_\theta, u_z] = [AK_1(q\gamma r)\cos\gamma(z-ct),\ 0,\ BK_0(q\gamma r)\sin\gamma(z-ct)]$$

where $K_1(q\gamma r)$, $K_0(q\gamma r)$ are modified Bessel functions of second kind of orders one and zero respectively, c is the phase-velocity, Values of u_r u_θ, u_z in equation (3) satisfy the equations of motion provided A and B satisfy the equations

$$\left.\begin{aligned}
&A(c_{11}q^2 + \varrho c^2 - c_{44}) - Bq(c_{13}+c_{44}) = 0\\
&B(c_{44}q^2 + \varrho c^2 - c_{33}) + Aq(c_{13}+c_{14}) = 0
\end{aligned}\right\} \tag{4}$$

from which, on elimination of $A:B$, we obtain

$$q^4-q^2\left\{\frac{c_{11}c_{33}-\varrho c^2(c_{12}+c_{44})-c^2{}_{13}-2c_{13}c_{44}}{c_{11}c_{44}}\right\}+\frac{(c_{44}-\varrho c^2)(c_{33}-\varrho c^2)}{c_{11}c_{44}}=0 \tag{5}$$

If q_1^2, q_2^2 be the roots of this equation (5), we take as general solution

$$\left.\begin{aligned} u_r &= \{A_1K_1(q_1\gamma r)+A_2K_1(q_2\gamma r)\}\cos\gamma(z-ct), \\ u_\theta &= 0 \\ u_z &= \left\{A_1\,\frac{c_{11}q_1^2+\varrho c^2-c_{44}}{q_1(c_{13}+c_{44})}\,K_0(q_1\gamma r)+A_2\,\frac{c_{11}q_2^2+\varrho c^2-c_{44}}{q_2(c_{13}+c_{44})}\,K_0(q_2\gamma r)\right\}\sin\gamma(z-ct) \end{aligned}\right\} \tag{6}$$

Phase Velocity Equation. In order that the surface of the cylindrical hole may be stressfree, we must have

$$\left.\begin{array}{lll} \widehat{rr}=0 & \text{on the surface} & r=a \\ \widehat{r\theta}=0 & \text{,,} & \text{,,} \\ \widehat{rz}=0 & \text{,,} & \text{,,} \end{array}\right\} \tag{7}$$

In order that stresses obtained from (6) may satisfy the boundary conditions (7), it is found that the phase velocity c satisfies the equation

$$\begin{aligned} &(c_{13}+c_{44})(c_{12}-c_{11})(q_2^2-q_1^2)\,\frac{c_{11}}{\gamma a} \\ &+\frac{1}{q_1}\frac{K_0(aq_1\gamma)}{K_1(aq_1\gamma)}\,[\{\varrho c^2-c_{44})c_{13}-c_{11}c_{44}q_1^2\}\{c_{11}q_2^2+\varrho c^2+c_{13}\}] \\ &-\frac{1}{q_2}\frac{K_0(aq_2\gamma)}{K_1(aq_2\gamma)}\,[\{(\varrho c^2-c_{43})c_{13}-c_{11}c_{44}q_2^2\}\{c_{11}q_1^2+\varrho c^2+c_{13}\}]=0 \end{aligned} \tag{8}$$

Equations (5) & (8) when simultaneously solved determine the phase velocity of the wave. Equation (8) shows that phase velocity c depends on wave length $(=2\pi/\gamma)$ of the wave, and therefore wave exhibits dispersion.

In the short wave length extreme, *i.e.* when the wave length $2\pi/\gamma$ is small compared to the radius of the hole, on using the asymptotic expansion of $K_n(aq\gamma)$, namely,

$$K_n(aq\gamma)\simeq\left(\frac{\pi}{2aq\gamma}\right)^{\frac{1}{2}}e^{-aq\gamma}\cos\,(n\,aq\gamma)\,[1+\dots]$$

for large value of $aq\gamma$, the equation (8) reduces to

$$\begin{aligned} &\frac{1}{q_1}\,[\{(\varrho c^2-c_{44})c_{13}-c_{11}c_{44}q_1^2\}\,\{c_{11}q_2^2+\varrho c^2+c_{13}\}] \\ &-\frac{1}{q_1}[\,\{(\varrho c^2-c_{44})c_{13}-c_{11}\,c_{44}\,q_2^2\}\{c_{11}q_1^2+\varrho c^2+c_{13}\}]=0 \end{aligned} \tag{9}$$

which when solved with the values of elastic constants as given by Voigt (Love Art 113) we obtain the value

$$c = 0{\cdot}90\sqrt{c_{44}/\varrho}$$

It is to be noted that in the isotropic case, on utilising the additional relations among elastic constants, namely,

$$c_{12} = c_{13}, \quad c_{11} = c_{33} = c_{12}+2c_{44},$$

we have

$$q_1^2 = 1-c^2/\alpha^2, \quad q_2^2 = 1-c^2/\beta^2,$$

α, β being velocities of compressional and equivoluminal waves respectively and then equation (8) reduces to that obtained by Biot (1952).

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ON SETS UNDER CERTAIN TRANSFORMATIONS

By

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Kestelman (1947) considered the following problem. Let E be a set of real numbers and $\{\lambda_n\}$ be any null sequence (*i.e.*, $\lambda_n \to 0$ as $n \to \infty$). Does there exist a single real number ξ such that $\xi + \lambda_n \in E$ for all large n? If there is, then he says that $\{\lambda_n\}$ 'belongs' to E. If the set E be such that every null sequence belongs to E, then he says that E is universal and writes $E \in u$. Now, every interval is certainly universal. He then sets out to discuss how much of an interval may be thrown out so that the remainder may still remain universal. He proves that if a set is universal then its distance set fills a continuum. But the converse is not necessarily true. This takes us to the very interesting result of Steinhaus (1920), *viz.* that the distance set of a set of positive interior measure fills a continuum $0 \leq x \leq \delta$ where $\delta (> 0)$ is a suitable positive number. Kestelman then sets out to generalize the results to a set E of positive interior measure in n-dimensional Euclidean space E_n and since every set of positive interior measure contains a closed set of positive measure, he chooses to confine his discussion to closed sets of positive measure.

Among other results he has shown in his main theorem that if C is a closed bounded set of positive measure in E_n and p be any positive integer, then there exists a positive number δ (depending on p and the measure of C) such that if $\lambda_1, \lambda_2, \ldots \lambda_p$ are any vector in E_n whose lengths $|\lambda_r|$ satisfy the relations $|\lambda_r| < \delta$, $r = 1, 2, \ldots p$; then it is always possible to find a closed set $C_1 \subset C$ of positive measure having the property that if $\xi \in C_1$ then $\xi + \lambda_r \in C$, $r = 1, 2, \ldots p$*. He has further shown that if $\{v_n\}$ be any null sequence of vectors, then there exists a subsequence $\{v_{n_r}\}$ of $\{v_n\}$ which belongs to C and he has proved that all null sequences which converges fast enough belongs to C.

Now, let C be any set in E_n and let T denote any general linear transformation of the form:

$$x'_i = \sum_{j=1}^{n} a_{ij} x_j + a_{i,\, n+1}, \quad i = 1, 2, \ldots n.$$

we shall denot by TC the set of points of E_n which are the transformed points of C under T.

Now, if $T_r (r = 1, 2, \ldots p)$ denote the transformation of translation, *viz.*,

$$x'_i = x_i + a_i^r, \quad i = 1, 2, \ldots n$$

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*For $n = 1$, $P = 1$, this implies Steinhaus's result.

then we see from the above considerations that Prof. Kestelman, in fact, proved that if C is any bounded closed set of positive measure in E_n, then there exists a positive number δ (depending on p and the measure of C) such that if $\lambda_r = (a_1{}^r, \ldots a_n{}^r)$ $r = 1, 2, \ldots p$ are vectors in E_n satisfying $|\lambda_r| < \delta$, then the set

$$C(T_1C)(T_2C)\ldots(T_pC) = X(\text{say})$$

is a bounded closed set of positive measure.

In a previous paper (Lahiri, 1959) I proved a number of results similar to those of Kestelman by considering the transformations T_r in the following manner

$$x_i' = b_r x_i, \quad i = 1, 2, \ldots n$$

where $b_r (\neq 0)$ satisfy certain conditions and, in particular, it has been shown that if C is a closed bounded set of positive measure in E_n, then under certain conditions on the elements b_r of T_r, the set X as defined above form a closed set of positive measure.

From the above discussions it seems reasonable for us to enquire whether the set X as considered above form a closed set of positive measure if $T_r (r = 1, 2, \ldots p)$ denotes any general linear transformation of the form

$$x'_i = \sum_{j=1}^{n} a^r_{ij} x_j + a^r_{i,\, n+1}, \quad i = 1, 2, \ldots n$$

where we assume that the determinant of the transformation viz.,

$$D_r = \begin{vmatrix} a^r_{11} \ldots a^r_{1n} \\ \ldots\ldots\ldots \\ a^r_{n1} \ldots a^r_{nn} \end{vmatrix}$$

is different from zero and $a^r{}_{ij}$ satisfy certain conditions.

Here, if $x = (x_1, \ldots x_n)$ is any point then we denote by x' the transformed point of x. Since $D_r \neq 0$, there exists a one-one correspondence between C and T_rC such that any point x of C corresponds uniquely to a point x' of T_rC under T_r and any x' of T_rC corresponds uniquely to x of C under $T_r{}^{-1}$.

In this connection we recall a well-known result. If C denote any set in E_n and T any linear transformation, then it is known that (Caratheodory. C., 1948)

$$m^*(TC) = |D|\, m^*(C)$$

and

$$m_*(TC) = |D|\, m_*(C)$$

where $|D|$ is the absolute value of the determinant of the transformation and $m^*(X)$, $m_*(X)$ denote the Lebesgue exterior and interior measue of X. So, if C is Lebesgue measurable, then TC is also so and

$$m(TC) = |D|\, m(C)$$

We shall henceforth understand by the term '*measure of a set*' its Lebesgue measure.

In this paper we obtain generalisations of some of the results of Prof. Kestelman taking for the transformation T_r (with suitable restrictions) any linear transformation with non-zero determinant and we present our results in the form of the following theorems:

Theorem 1. *Let C be a bounded closed set of positive measure in E_n and p be any positive integer. Then we can find a positive number M and a number $\delta(>0)$ depending on p and the measure of C such that if T_r, $r = 1, 2, \ldots p$ are any linear transformations given by*

$$x'_i = \sum_{j=1}^{n} a^r_{ij} x_j + a^r_{i,\,n+1}, \quad i = 1, 2, \ldots n$$

where

$$1 \leq a^r_{ij} < 1 + \frac{\delta}{(M+1)n}, \quad i = j$$

$$0 \leq a^r_{ij} < \frac{\delta}{(M+1)n}, \quad i \neq j$$

with non-zero determinants, then the set of points x such that

$$x \in C \text{ and } T_r^{-1} x \in C, \qquad (r = 1, 2, \ldots p)$$

form a closed set of positive measure.

Proof. Since C is bounded, there exists a sphere of radius, say M containing C. Now, let U be an open set containing C and itself contained in the sphere of radius M satisfying the relation

$$m(U) - m(C) < m(C)\,\frac{2p-1}{8p(p+1)} \tag{1}$$

$m(S)$ denote the Lebesgue measure of a measurable set S, and if $\delta(>0)$ be the distance of C and U' [Complement of U], then we may easily choose the open set U in such a manner that

$$\delta < \frac{1}{8p} \cdot \frac{1}{2^n n!} \tag{2}$$

Now, a point ξ of C transforms to a point ξ' under the transformation T_r. The set of ξ''s constitute a closed set C_r, say whose measure is given by

$$m(C_r) = |D_r|\, m(C)$$

[where $|D_r|$ denotes the absolute value of the determinant of the transformation T_r. Since $D_r \neq 0$, $r = 1, 2, \ldots p$, the measure of C_r is positive.

Now, if X' is the corresponding point of a point X of C under T_r, then from the conditions on the elements a^r_{ij}, it follows that

$$|X' - X| < \delta$$

($|x-y|$ denote the distance between x and y)

So, $$C_r \subset U \text{ for } r = 1, 2, \ldots p$$

Now, $$D_r = \begin{vmatrix} a^r_{11} \ldots a^r_{1n} \\ \ldots\ldots\ldots \\ a^r_{n1} \ldots a^r_{nn} \end{vmatrix}$$

It can be easily seen that under the conditions on the elements $a^r{}_{ij}$

$$D_r > 1 - \Sigma a_{i_1j_1} a_{i_2j_2} \ldots a_{i_nj_n}$$

where the summation contains $\frac{1}{2}n!$ terms.

Again, one can easily verify that any term

$$a_{i_1j_1} a_{i_2j_2} \ldots a_{i_nj_n}$$

of the above summation satisfy the relation

$$a_{i_1j_1} a_{i_2j_2} \ldots a_{i_nj_n} < \left(1 + \frac{\delta}{(M+1)n}\right)^{n-2} \cdot \frac{\delta}{(M+1)n} \cdot \frac{\delta}{(M+1)n} \cdot$$

So, $$D_r > 1 - \frac{n!}{2}\left(1 + \frac{\delta}{(M+1)n}\right)^{n-2} \cdot \frac{\delta^2}{(M+1)^2 n^2}$$

$$> 1 - 2^n . n! \delta$$

$$> 1 - \frac{1}{3p} \text{ from } (2)$$

So, $$m(Cr) = |D_r| \, m(C) > \left(1 - \frac{1}{3p}\right) m(C) > 0$$

for $$r = 1, 2, \ldots, p$$

Now, let Y denote the set $C_0 C_1 C_2 \ldots C_p$, $C_0 = C$

Then $$Y = \sum_{r=0}^{p} C_r$$

$$= U - \sum_{r=0}^{p} (U - C_r)$$

Therefore $$m(Y) \geq m(U) - \sum_{r=0}^{p} \{m(U) - m(C_r)\}$$

$$> m(U) - (p+1)\left\{m(U) - \left(1 - \frac{1}{3p}\right) m(C)\right\}$$

Again, from (1) $$m(U) - m(C)\left(1 - \frac{1}{3p}\right) < \frac{m(C)}{p+1}$$

So,
$$m(Y) > m(U)-(p+1)\frac{m(C)}{p+1}$$
$$= m(U)-m(C) > 0$$

Hence the set of points x common to the sets C, C_1, $C_2, \ldots C_p$ form a set of positive measure.

Now, since each C_i is closed, so Y is closed and if $x \in Y$, then

$$x \in C_1,\ C_2, \ldots C_p,\ C$$

and $x \in C_r$ implies $T_r^{-1} x \in C$

So, Y is the set of points with the required property. This proves the theorem

Theorem 2. *Let C be a closed set of positive measure contained in an open sphere of radius, say M in the n-dimensional Euclidean space and let $T_{\delta r}$ denote the linear transformation*

$$x_i' = \sum_{j=1}^{n} a_{ij}^r x_j + a_{i,\,n+1}^r, \quad i = 1, 2, \ldots n$$

with non zero determinant where the elements a^r_{ij} satisfy the relations

$$a_{ij}^r = 1 + \frac{\delta_r}{2(M+1)n}, \quad i = j$$

$$a_{ij}^r = \frac{\delta_r}{2(M+1)n}, \quad i \neq j$$

Then, if $\{\lambda_r\}$, $\lambda_r > 0$ be any null sequence, there exists a sub-sequence $\{\lambda_{n_r}\}$ of $\{\lambda_r\}$ and a point X of C such that

$$T_{\lambda_{n_r}}^{-1} X \in C, \quad r = 1, 2, \ldots$$

Proof. Since C is a closed set of positive measure, we get from Theorem 1 that there exists a positive integer n_1 such that if $T_{\lambda_{n_1}}$ is the corresponding transformation, then the set of points X such that

$$X \in C \quad \text{and} \quad T_{\lambda_{n_1}}^{-1} X \in c$$

form a closed set of positive measure. Let us denote this set by C_1.

So,
$$C_1 \subset C,\ \bar{C}_1 = C_1,\ m(C_1) > 0$$

Again, since C_1 is a closed set of positive measure, we can find a positive integer $n_2 (> n_1)$ such that the set of points X such that

$$X \in C_1 \quad \text{and} \quad T_{\lambda_{n_2}}^{-1} X \in C_1$$

form a closed set of positive measure. Let this set be denoted by C_2.

So, $$C_2\subset C_1\subset C,\ \bar{C}_2 = C_2,\ m(C_2)\geq 0$$

We again apply the same procedure on C_2 and obtain a set C_3 with the property that

$$C_3\subset C_2\subset C_1\subset C,\ \bar{C}_3 = C_3,\ m(C_3) > 0$$

and C_3 is the set of points X such that

$$X\in C_2 \text{ and } T^{-1}_{\lambda_{n_3}}X\in C_2$$

This process is continued indefinitely.

So, we obtain sets $C_1, C_2, C_3, \dots$ such that

$$C\supset C_1\supset C_2\supset C_3\supset\dots$$

and a sequence of positive integers $\{n_r\}$ such that

$$\lambda_{n_1} > \lambda_{n_2} > \lambda_{n_3} > \dots$$

Since all the sets C_i are closed and non-empty, so the product is non-empty and so there exists at least one point $X\in\prod_{i=0}^{\infty} C_i,\ C_0 = C$

Now, $$X\in C_r \qquad \text{implies } T^{-1}_{\lambda_{n_r}}X\in C_{r-1}\subset C$$

for $$r = 1, 2, \dots$$

Hence there exists a point $X\in C$ such that

$$T^{-1}_{\lambda_{n_1}}X\in C,\ T^{-1}_{\lambda_{n_2}}X\in C, \dots T^{-1}_{\lambda_{n_r}}X\in C, \dots$$

This proves the theorem.

Theorem 3. *Let C be a closed set of positive measure contained in an open sphere of radius, say M in the n-dimensional Euclidean space. Then we can find a positive null sequence $\{\mu_r\}$ such that if $\{\delta_r\}$ be any null sequence satisfying*

$$0 < \delta_r < \min\left\{\mu_r, \frac{1}{2^{n+r+1}n\,!}\right\}$$

and T_{δ_r} is the linear transformation

$$x_i' = \sum_{j=1}^{n} a^r_{ij}x_j + a^r_{i,\,n+1}, \quad i = 1, 2, \dots n$$

with non-zero determinant where

$$a^r_{ij} = 1 + \frac{\delta_r}{2(M+1)n}, \quad i = j$$

$$a^r_{ij} = \frac{\delta_r}{2(M+1)n}, \qquad i\neq j$$

then the set of points X such that

$$X \in C \quad \text{and} \quad T_{\delta_r}^{-1} X \in C, \quad r = 1, 2, \ldots$$

form a closed set of positive measure.

Proof. As shown in Kestelman's paper (1947), we can define a sequence of open sets $\{u_r\}$ such that

$$U_1 \supset U_2 \supset U_3 \ldots ; \; C = \prod_{r=1}^{\infty} U_r$$

and

$$\sum_{r=1}^{\infty} \{m(U_r) - m(C)\} < \frac{m(C)}{2} \tag{4}$$

Let μ_r (> 0) be the distance between C and U'_r [Complement of U_r]. Then evidently $\{\mu_r\}$ is a positive null sequence. We choose the number δ_r (> 0) to be less than the lesser of the numbers μ_r and $\frac{1}{2^{n+r+1} n\,!}$. Then, obviously $\{\delta_r\}$ is also a positive null sequence. For each δ_r we consider the corresponding transformation T_{δ_r}. Let C_r be the set of points which are the transform of the points of C under the transformation T_{δ_r}.

i.e.,

$$C_r = T_{\delta_r} C$$

Now, it can be easily shown as in Theorem 1 that $C_r \subset U_r$ for every r.

A point X of C transforms under T_{δ_r} to a point X' of C_r such that

$$|X' - X| < \delta_r < \mu_r.$$

Let

$$Z = C_1 C_2 C_3 \ldots C_r \ldots$$

Then, if

$$\xi \in Z$$

$$\xi \in C_1, \; \xi \in C_2, \; \xi \in C_3, \ldots \xi \in C_r, \ldots$$

i.e.,

$$T_{\delta_1}^{-1} \xi \in C, \; T_{\delta_2}^{-1} \xi \in C, \; T_{\delta_3}^{-1} \xi \in C, \ldots T_{\delta_r}^{-1} \xi \in C, \ldots$$

and

$$|\xi - T_{\delta_r}^{-1} \xi| < \delta_r$$

So,

$$\underset{r \to \infty}{\mathrm{Lt}} \; T_{\delta_r}^{-1} \xi = \xi \qquad \because \quad \delta_r \to 0.$$

Since C is closed, so $\xi \in C$.

Hence Z is the set of points ξ of C such that

$$T_{\delta_1}^{-1} \xi \in C, \; T_{\delta_2}^{-1} \xi \in C, \ldots T_{\delta_r}^{-1} \xi \in C, \ldots$$

Again, since each C_r is closed, so Z is closed. So, we are only to show that the measure of Z is positive.

Now,

$$\begin{aligned} Z &= C_1 C_2 C_3 \ldots \\ &= U_1 - \sum_{r=1}^{\infty} (U_1 - C_r) \\ &= U_1 - \Sigma(U_1 - U_r + U_r - C_r) \\ &= C - \Sigma(U_r - C_r); \end{aligned}$$

Therefore $$m(Z) \geq m(C) - \Sigma\{m(U_r) - m(C_r)\}$$

Now, we can easily show following the method as adopted in the proof of Theorem 1 that if D_{δ_r} denote the determinant of the transformation T_{δ_r} then

$$D_{\delta_r} > 1 - \delta_r(2^n.n!)$$

$$> 1 - \frac{1}{2^{r+1}}, \quad r = 1, 2, \ldots$$

So, $$m(C_r) = D_{\delta_r} m(C) > m(C) - \frac{m(C)}{2^{r+1}}$$

and therefore $$m(U_r) - m(C_r) < m(U_r) - m(C) + \frac{m(C)}{2^{r+1}};$$

Therefore $$m(Z) > m(C) - \Sigma\left\{m(U_r) - m(C) + \frac{m(C)}{2^{r+1}}\right\}$$

$$= \frac{m(C)}{2} - \Sigma\{m(U_r) - m(C)\}$$

$$> 0 \qquad \text{from (4).}$$

Hence Z is the set of points with the required property. This proves the theorem.

The results of the theorems 2 and 3 will hold true even if by T_{δ_r} we mean any particular linear transformation of the whole class of transformations given by

$$x_i' = \sum_{j=1}^{n} a_{ij}^r x_j + a_{i,\,n+1}^r, \quad i = 1, 2, \ldots n$$

where $$1 \leq a_{ij}^r < 1 + \frac{\delta_r}{(M+1)n}, \quad i = j$$

$$0 \leq a_{ij}^r < \frac{\delta_r}{(M+1)n}, \quad i \neq j$$

Now, let S be any set of points in the n-demensional Euclidean space and $\{\delta_r\}$ be any positive null sequence. Then one may enquire whether it is always possible to find a point ξ such that

$$T_{\delta_r}^{-1}\xi \in S \text{ for all large } r.$$

As observed by Kestelman if S be an open interval, then such a ξ exists. In fact every point ξ of S has this property, since $|\xi - T_{\delta_r}^{-1}\xi| < \delta_r$.

Now, if a set S has the property that for any null sequence $\{\delta_r\}$, we can find a point ξ such that

$$T_{\delta_r}^{-1}\xi \in S$$

for all large r, then we say that S possesses the property P and, in notation, we write $S \in P$.

So, intervals possess the property P. We now enquire, after Kestelman, what sets may be removed from an interval such that the remaining set still possesses the property P.

Here also we prove results parallel to those of Kestelman's, viz that if I be an interval and Z a subset of I, then $I-Z\epsilon P$, if (*a*) Z is of measure zero, (*b*) Z is a set of the first category.

Theorem 4. *Let I be an open interval in E_n and Z be any subset of I of measure zero. Then for any positive null sequence $\{\delta_r\}$, we can find a point ξ such that*

$$T_{\delta_r}^{-1}\xi\epsilon I-Z \quad \textit{for all large } r.$$

Proof. Let $C_r = T_{\delta_r}Z$, then $m\,(C_r) = 0$

Again, let
$$X = I\sum_{r=1}^{\infty} C_r$$

Since the measure of each C_r is zero, so the measure of X is zero.

Now, let
$$\xi\epsilon I-X$$
i.e.,
$$\xi\epsilon I,\ \xi \text{ not } \epsilon C_1,\ C_2,\ C_3,\dots,\ \dots$$
i.e.,
$$T_{\delta_1}^{-1}\xi \text{ not } \epsilon Z,\ T_{\delta_2}^{-2}\xi \text{ not } \epsilon Z,\ T_{\delta_3}^{-1}\xi \text{ not } \epsilon Z,\dots$$
Since
$$\xi\epsilon I \quad\text{and}\quad |T_{\delta_r}^{-1}\xi-\xi|<\delta_r$$
so
$$T_{\delta_r}^{-1}\xi\epsilon I-Z \qquad \text{for all large } r.$$

Consequently, it has been proved also that for almost all points ξ of I, $T_{\delta_r}^{-1}\xi\epsilon I-Z$ for all large r.

we may likewise prove the

Theorem 5. *Let I be an open interval in E_n and $Z\subset I$ be any set of the first category. Then for any positive null sequence $\{\delta_r\}$ it is possible to find a residual set $R\subset I$ such that for any point $\xi\epsilon R$*

$$T_{\delta_r}^{-1}\xi\epsilon I-Z \qquad \textit{for all large } r.$$

If in the above theorems we consider the transformations in such a manner that

$$a^r_{ij} = 1 \quad \text{for} \quad i = j$$
$$a^r_{ij} = 0 \quad \text{for} \quad i\neq j,\ \ j\neq n+1$$
$$0\leq a^r_{i,\,n+1} < \frac{\delta}{(M+1)n}, \quad i = 1,\ 2,\dots n$$

then the above results coincide with the corresponding results of Prof. Kestelman.

In conclusion, I offer my sincere gratitude to Dr. H. M. Sengupta for his kind help in the preparation of the paper.

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SOME PROBLEMS OF PLANE STRESS IN AN AEOLOTROPIC CIRCULAR DISK

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Introduction. The problem of finding stresses in an aeolotropic circular disk have been investigated by several authors. (H. F. Cleaves, 1955, A. M. Sen Gupta, 1949) An approximate solution has been obtained (Love, pp. 147-148) in the case of a thin disk of thickness $2l$ by treating the problem as one of plane stress. If the disk has no circular axle-hole the solution satisfies $\widehat{zz} = 0$ $\widehat{rz} = 0$ and $\int_{-l}^{+l} \widehat{rr}\, dz = 0$ for $r = a$. Also the displacements U and w vanish at the origin. $(r = 0,\ z = 0)$ When there is a circular axle-hole of radius b the solution satisfies the equations $\widehat{zz} = 0$, $\widehat{rz} = 0$ and $\int_{-l}^{+l} \widehat{rr}\, dz = 0$ when $r = a$ and $r = b$.

In this paper general expressions for the displacements and stress components both for the aeolotropic and hexagonally aeolotropic circular disk are obtained. By giving suitable values to the elastic constants solutions given in Love (1952, pp. 147-148) have been deduced as particular cases.

In cylindrical co-ordinates (r, θ, z) the stress-strain relations can be taken in the form

$$
\begin{gathered}
(\widehat{rr}, \widehat{\theta\theta}, \widehat{zz}) = \begin{pmatrix} c_{11}, & c_{12}, & c_{13} \\ c_{21}, & c_{22}, & c_{23} \\ c_{31}, & c_{32}, & c_{33} \end{pmatrix} (e_{rr}, e_{\theta\theta}, e_{zz}) \\
\widehat{\theta z} = c_{44} e_{\theta z}, \quad \widehat{rz} = c_{55} e_{rz}, \quad \widehat{r\theta} = c_{66} e_{r\theta}
\end{gathered}
\tag{1}
$$

and $c_{rs} = c_{sr}$ which is the condition for the existence of the strain-energy-function. The conditions that the displacements may take place in planes through an axis be the same in all such planes would be expressed by the equations

$$u_\theta = 0, \quad \frac{\partial u_r}{\partial \theta} = 0, \quad \frac{\partial u_z}{\partial \theta} = 0$$

where u_r, u_θ, u_z are the displacements. Let $u_r = U$ and $u_z = w$. Then the strain-components are given by

$$
\left.
\begin{aligned}
e_{rr} &= \frac{\partial U}{\partial r}, \quad e_{\theta\theta} = \frac{U}{r}, \quad e_{zz} = \frac{\partial w}{\partial z} \\
e_{rz} &= \frac{\partial U}{\partial z} + \frac{\partial w}{\partial r}, \quad e_{r\theta} = e_{\theta z} = 0
\end{aligned}
\right\}
\tag{2}
$$

The stress-equations in cylindrical coordinates are given by (Love, 1952, pp. 90)

$$\left.\begin{aligned}
\frac{\partial\widehat{rr}}{\partial r}+\frac{1}{r}\frac{\partial\widehat{r\theta}}{\partial\theta}+\frac{\partial\widehat{rz}}{\partial z}+\frac{\widehat{rr}-\widehat{\theta\theta}}{r} &= \varrho(f_r-F_r)\\
\frac{\partial\widehat{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\widehat{\theta\theta}}{\partial\theta}+\frac{\partial\widehat{\theta z}}{\partial z}+\frac{2\widehat{r\theta}}{r} &= \varrho(f_\theta-F_\theta)\\
\frac{\partial\widehat{rz}}{\partial r}+\frac{1}{r}\frac{\partial\widehat{\theta z}}{\partial\theta}+\frac{\partial\widehat{zz}}{\partial z}+\frac{\widehat{rz}}{r} &= \varrho(f_z-F_z)
\end{aligned}\right\} \tag{3}$$

For plane stress we must have $\widehat{zz}=0$, $\widehat{rz}=0$, $F_z=0$. We have to put $f_r=-\Omega^2 r$ where Ω is the angular velocity of the disk. The last two equations of (3) are identically satisfied. The first equation takes the form

$$\frac{\partial\widehat{rr}}{\partial r}+\frac{\widehat{rr}-\widehat{\theta\theta}}{r}=-\Omega^2\varrho r \tag{4}$$

from the conditions $\widehat{zz}=0$, $\widehat{rz}=0$ we deduce

$$c_{33}\frac{\partial w}{\partial z}=-\left(c_{31}\frac{\partial U}{\partial r}+c_{32}\frac{U}{r}\right),\quad \frac{\partial w}{\partial r}=-\frac{\partial U}{\partial z} \tag{5}$$

substituting

$$\left.\begin{aligned}
\widehat{rr} &= c_{11}\frac{\partial U}{\partial r}+c_{12}\frac{U}{r}+c_{13}\frac{\partial w}{\partial z}\\
\widehat{\theta\theta} &= c_{21}\frac{\partial U}{\partial r}+c_{22}\frac{U}{r}+c_{23}\frac{\partial w}{\partial z}
\end{aligned}\right\} \tag{6}$$

in (4) we obtain the following differential equation.

$$r^2\frac{\partial^2 U}{\partial r^2}+r\frac{\partial U}{\partial r}-\alpha^2 U=-\frac{\Omega^2\varrho c_{33}}{(c_{11}c_{33}-c^2_{13})}r^3 \tag{7}$$

where

$$\alpha^2=\frac{(c_{22}c_{33}-c^2_{23})}{(c_{11}c_{33}-c^2_{23})}$$

The complete solution of (7) is given by

$$U=Ar^{\alpha}+Br^{-\alpha}-\frac{\Omega^2\varrho c_{33}}{9(c_{11}c_{33}-c^2_{13})-(c_{22}c_{33}-c^2_{23})}r^3 \tag{8}$$

We impose the condition that (U, w) vanish at the origin $(r=0,\ z=0)$. In this case we take $B=0$ and the equation (8) reduces to

$$U=Ar^{\alpha}+B\,r^3 \tag{9}$$

where

$$B'=-\frac{\Omega^2\varrho c_{33}}{9(c_{11}c_{33}-c^2_{13})-(c_{22}c_{33}-c^2_{23})}$$

substituting equation (9) in (5) we find

$$\frac{\partial^2 U}{\partial z^2} = LAr^{\alpha-2} + Mr \tag{10}$$

where

$$L = \frac{(\alpha-1)(c_{31}\alpha + c_{32})}{c_{33}}, \quad M = \frac{2B'(3c_{31}+c_{32})}{c_{33}}$$

Hence we take U in the form

$$U = Ar^{\alpha} + B'r^3 + \tfrac{1}{2}(LAr^{\alpha-2} + Mr)Z^2 \tag{11}$$

The constant of integration A can be easily determined by using the relation

$$\int_{-l}^{+l} \widehat{rr}\, dz = 0 \text{ when } r = a. \tag{12}$$

Also

$$\widehat{rr} = N\frac{\partial U}{\partial r} + N'\frac{U}{r} \tag{13}$$

where

$$N = \frac{(c_{11}c_{33} - c^2_{13})}{c_{33}}, \quad N' = \frac{(c_{12}c_{33} - c_{13}c_{32})}{c_{33}}$$

with the help of (11) (12) (13) we obtain

$$A = -\frac{6B'(3N+N')a^2 + M(N+N')l^2}{6(\alpha N+N')a^2 + L(\alpha N+N'-2N)l^2} \cdot \frac{1}{a^{\alpha-3}} \tag{14}$$

Hence we have

$$\left.\begin{aligned} U &= Ar^{\alpha-2}\left(r^2 + \frac{Lz^2}{2}\right) + r\left(B'r^2 + \frac{Mz^2}{2}\right) \\ w &= -Ar^{\alpha-1}\left(\frac{c_{31}\alpha + c_{32}}{c_{33}}\right)z - \frac{L}{6}r^{\alpha-3}(c_{31}\alpha + c_{32} - 2c_{31})z^3 \\ &\quad - B'r^2\left(\frac{3c_{31}+c_{32}}{c_{33}}\right)z - \frac{M}{6}\left(\frac{c_{31}+c_{32}}{c_{33}}\right)z^3 \end{aligned}\right\} \tag{15}$$

From these equations we deduce the following expressions for the stress-components $\widehat{rr}$, $\widehat{\theta\theta}$ in the form

$$\left.\begin{aligned} \widehat{rr} &= A\left[r^{\alpha-1}(\alpha N + N') + \frac{L}{2}z^2 r^{\alpha-3}(\alpha N + N' - 2N)\right] + B'r^2(3N+N') + \tfrac{1}{2}Mz^2(N+N') \\ \widehat{\theta\theta} &= A\left[r^{\alpha-1}(\alpha N' + \alpha^2 N) + \frac{L}{2}z^2 r^{\alpha-3}(\alpha N' + \alpha^2 N - 2N')\right] + B'r^2(3N' + \alpha^2 N) + \tfrac{1}{2}Mz^2(N' + \alpha^2 N) \end{aligned}\right\} \tag{16}$$

2. When there is a circular axle-hole of radius b we have the additional condition $\int_{-l}^{+l} \widehat{rr}dz = 0$ for $r = b$. In this case we take U in the form

$$U = Ar^{\alpha}+Br^{-\alpha}+B'r^{3} \tag{17}$$

where A and B are constants of integration. Using the equations $\widehat{zz} = 0$, $\widehat{rz} = 0$ we find

$$\frac{\partial^2 U}{\partial z^2} = LAr^{\alpha-2}+Mr+QBr^{-(\alpha+2)} \tag{18}$$

where
$$Q = (\alpha+1)\frac{(\alpha(c_{31}-c_{32}))}{c_{33}}$$

from (17) and (18) we have

$$U = Ar^{\alpha}+Br^{-\alpha}+B'r^{3}+\tfrac{1}{2}(LAr^{\alpha-2}+Mr+QBr^{-(\alpha+2)})z^{2} \tag{19}$$

Using (13) (19) and $\int_{-l}^{+l} \widehat{rr}dz = 0$ for $r = a$ and $r = b$, we obtain after a little simplification the following equations from which the values of A and B can be obtained.

$$\left.\begin{aligned} Ap_1+Bp_2+p_3 &= 0 \\ Aq_1+Bq_2+q_3 &= 0 \end{aligned}\right\} \tag{20}$$

where
$$\left.\begin{aligned} p_1 &= 6a^{\alpha-1}(\alpha N+N')+La^{\alpha-3}(\alpha N+N'-2N)l^2 \\ p_2 &= 6a^{-(\alpha+1)}(N'-\alpha N)+Qa^{-(\alpha+3)}(N'-\alpha N-2N)l^2 \\ p_3 &= 6B^1a^2(3N+N')+M(\mathrm{N}+N')l^2 \\ q_1 &= 6b^{\alpha-1}(\alpha N+N')+Lb^{\alpha-3}(\alpha N+N'-2N)l^2 \\ q_2 &= 6b^{-(\alpha+1)}(N'-\alpha N)+Qb^{-(\alpha+3)}(N'-\alpha N-2N)l^2 \\ q_3 &= 6B^1b^2(3N+N')+M(N+N')l^2 \end{aligned}\right\} \tag{21}$$

The stress-components $\widehat{rr}$ and $\widehat{\theta\theta}$ are given by

$$\begin{aligned} \widehat{rr} &= Ar^{\alpha-1}(\alpha N+N')+Br^{-(\alpha+1)}(N'-\alpha N)+B'r^2(3N+N') \\ &\quad +\tfrac{1}{2}[LAr^{\alpha-3}(\alpha N+N'-2N)+M(N+N')+QBr^{-(\alpha+3)}(N'-\alpha N-2N]z^2 \\ \widehat{\theta\theta} &= Ar^{\alpha-1}\alpha(\alpha N+N')+Br^{-(\alpha+1)}\alpha(\alpha N-N')+B'r^2(\alpha^2N+3N') \\ &\quad +\tfrac{1}{2}[LAr^{\alpha-3}(\alpha^2N+\alpha N'-2N')+M(\alpha^2N+N')+QBr^{-(\alpha+3)}(\alpha^2N-\alpha N'-2N')]z^2 \end{aligned} \tag{22}$$

The displacement w is given by

$$\begin{aligned} c_{33}w &= -[Ar^{\alpha-1}(\alpha c_{31}+c_{32})+Br^{-(\alpha+1)}(c_{32}-\alpha c_{31})+B'r^3(3c_{31}+c_{32})]z \\ &\quad -\tfrac{1}{6}[LAr^{\alpha-3}(c_{31}+c_{32}-2c_{31})+M(c_{31}+c_{32})+QBr^{-(\alpha+3)}(c_{32}-\alpha c_{31}-2c_{31})]z^3 \end{aligned} \tag{23}$$

3. Writing $c_{11} = c_{22} = c_{33} = (\lambda+2\mu)$ and $c_{12} = c_{31} = c_{23} = \lambda_1$, $\sigma = \dfrac{\lambda}{2(\lambda+\mu)}$ we find

$$\left.\begin{aligned}
&L = 0, \quad Q = 0, \quad \alpha = 1. \quad N = \frac{2\mu}{1-\sigma} \qquad N' = \frac{2\mu\sigma}{1-\sigma}\\
&M = -\frac{\Omega^2\varrho\sigma}{2\mu} \qquad B' = -\frac{\Omega^2\varrho(1-\sigma}{16\mu} \quad \text{and } E = 2\mu(1+\sigma)\\
&p_1 = 12\mu\left(\frac{1+\sigma}{1-\sigma}\right),\ p_2 = -\frac{12\mu}{a^2},\ p_3 = -\tfrac{3}{4}\Omega^2\varrho a^2(3+\sigma) - \Omega^2\varrho\sigma\left(\frac{1+\sigma}{1-\sigma}\right)l^2\\
&q_1 = 12\mu\left(\frac{1+\sigma}{1-\sigma}\right),\ q_2 = -\frac{12\mu}{b^2},\ q_3 = -\tfrac{3}{4}\Omega^2\varrho b^2(3+\sigma) - \Omega^2\varrho\sigma\left(\frac{1+\sigma}{1-\sigma}\right)l^2
\end{aligned}\right\} \tag{24}$$

The value of A given by (14) reduces to

$$A = \frac{\Omega^2\varrho(1-\sigma)(3+\sigma)a^2}{8E} + \frac{\Omega^2\varrho\sigma(1+\sigma)l^2}{8E} \tag{25}$$

and the displacements and the stress components are now given by

$$\left.\begin{aligned}
U &= \frac{\Omega^2\varrho r(1-\sigma)}{8E}\left[(3+\sigma)a^2 - (1+\sigma)r^2\right] + \frac{\Omega^2\varrho r}{6E}\sigma(1+\sigma)(l^2-3z^2)\\
w &= -\frac{\Omega^2\varrho z}{4E}\sigma\left[(3+\sigma)a^2 - 2(1+\sigma)r^2\right] - \frac{\Omega^2\varrho z}{3E}\sigma^2\left(\frac{1+\sigma}{1-\sigma}\right)(l^2-z^2)
\end{aligned}\right\} \tag{26}$$

$$\left.\begin{aligned}
\widehat{rr} &= \frac{\Omega^2\varrho}{8}(3+\sigma)(a^2-r^2) + \frac{\Omega^2\varrho\sigma}{6}\left(\frac{1+\sigma}{1-\sigma}\right)(l^2-3z^2)\\
\widehat{\theta\theta} &= \frac{\Omega^2\varrho}{8}\left[(3+\sigma)a^2 - (1+3\sigma)r^2\right] + \frac{\Omega^2\varrho\sigma}{6}\left(\frac{1+\sigma}{1-\sigma}\right)(l^2-3z^2)
\end{aligned}\right\} \tag{27}$$

solving the equations (20) and using (24) we find

$$\left.\begin{aligned}
&A = \frac{\Omega^2\varrho(1-\sigma)(3+\sigma)(a^2+b^2)}{8E} + \frac{\Omega^2\varrho\sigma(1+\sigma)l^2}{6E}\\
\text{and} \quad &B = \frac{\Omega^2\varrho(1+\sigma)(3+\sigma)a^2b^2}{8E}
\end{aligned}\right\} \tag{28}$$

In this case the displacements are given by

$$\begin{aligned}
U = {} & \frac{\Omega^2\varrho(1-\sigma)r}{8E}\left[(3+\sigma)a^2 - (1+\sigma)r^2\right] + \frac{\Omega^2\varrho\sigma}{6E}r(l^2-3z^2)(1+\sigma)\\
& + \frac{\Omega^2\varrho}{8E}(3+\sigma)r\left[(1-\sigma)b^2 + \frac{(1+\sigma)a^2b^2}{r^2}\right]
\end{aligned} \tag{29}$$

$$w = -\frac{\Omega^2 \varrho z}{4E}\sigma[(3+\sigma)a^2 - 2(1+\sigma)r^2] - \frac{\Omega^2 \varrho z}{3E}\sigma^2\left(\frac{1+\sigma}{1-\sigma}\right)(l^2 - z^2) - \frac{\Omega^2 \varrho\sigma(3+\sigma)b^2 z}{4E}$$

The stress components are given by

$$\widehat{rr} = \frac{\Omega^2 \varrho}{8}(3+\sigma)(a^2 - r^2) + \frac{\Omega^2 \varrho\sigma}{6}\left(\frac{1+\sigma}{1-\sigma}\right)(l^2 - 3z^2) + \frac{\Omega^2 \varrho}{8}(3+\sigma)\left(b^2 - \frac{a^2 b^2}{r^2}\right)$$

$$\widehat{\theta\theta} + \frac{\Omega^2 \varrho}{8}[(3+\sigma)a^2 - (1+3\sigma)r^2] + \frac{\Omega^2 \varrho\sigma}{6}\left(\frac{1+\sigma}{1-\sigma}\right)(l^2 - 3z^2) + \frac{\Omega^2 \varrho}{8}(1+3\sigma)\left(b^2 + \frac{a^2 b^2}{r^2}\right) \quad (30)$$

These results agree with the results given in Love (1952, pp. 148).

In case of curvilinear aeolotropy we write

$$c_{11} = d_1,\ c_{22} = c_{33} = d_2,\ c_{13} = c_{31} = c_{12} = c_{21} = d_3$$

$$c_{23} = c_{32} = d_4 \quad \text{etc.}$$

For hexagonally aeolotropic disk we have

$$\alpha = 1,\ L = 0,\ Q = 0$$

$$\left.\begin{aligned}
N &= \frac{c_{11}c_{33} - c_{13}^2}{c_{33}}, \quad N' = \frac{c_{12}c_{33} - c_{13}^2}{c_{33}}, \quad B' = -\frac{\Omega^2 \varrho}{8N},\\
M &= -\frac{\Omega^2 \varrho c_{13}}{N' c_{33}}, \quad p_1 = 6(N+N'), \quad p_2 = \frac{6(N'-N)}{a^2},\\
p_3 &= -\frac{3\Omega^2 \varrho a^2}{4N}(3N+N') - \frac{\Omega^2 \varrho}{N}\cdot\frac{c_{31}}{c_{33}}(N+N')l^2,\\
q_1 &= 6(N+N'), \quad q_2 = \frac{6(N'-N)}{b^2},\\
q_3 &= -\frac{3\Omega^2 \varrho b^2}{4N}(3N+N') - \frac{\Omega^2 \varrho}{N}\cdot\frac{c_{31}}{c_{33}}(N+N')l^2,
\end{aligned}\right\} \quad (32)$$

$$A = \frac{\Omega^2 \varrho}{24N(N+N')}\left[3(3N+N')a^2 + \frac{4c_{31}}{c_{33}}(N+N')l^2\right], \quad (33)$$

The displacements and stress components are expressed by the following equations.

$$\left.\begin{aligned}
U &= \left(A + \frac{M.z^2}{2}\right)r + B'r^3\\
w &= -\frac{2c_{31}}{c_{33}}\left(A.z + \frac{M z^3}{6} + 2B'.r^2.z\right)
\end{aligned}\right\} \quad (34)$$

$$\left.\begin{aligned}\widehat{rr} &= \left(A+\frac{M.z^2}{2}\right)(N+N')+(3N+N')B'.r^2\\ \widehat{\theta\theta} &= \left(A+\frac{M.z^2}{2}\right)(N+N')+(3N'+N)B'.r^2\end{aligned}\right\}\qquad(35)$$

where A, B', M, N, N' are given by (32) and (33)

For a circular disk having a circular axle-hole the constants A and B are determined from (20). Also p_i, $q_i (i = 1, 2, 3)$ are given by (32)

$$A = \frac{\Omega^2\varrho}{6N(N+N')}\left[\tfrac{3}{4}(3N+N')(a^2+b^2)+\frac{c_{31}}{c_{33}}\,l^2\right]$$

$$B = \frac{\Omega^2\varrho}{8N(N-N')}(3N+N')a^2b^2 \qquad (36)$$

The displacements and the Stress Components are given by

$$\left.\begin{aligned}U &= (A+\tfrac{1}{2}Mz^2)r+B'r^3+B/r\\ w &= -2c_{31}/c_{33}(Az+2B'r^2z+\tfrac{1}{6}Mz^3)\end{aligned}\right\}\qquad(37)$$

$$\left.\begin{aligned}\widehat{rr} &= (N+N')(A+\tfrac{1}{2}Mz^2)+B'r^2(3N+N')+B/r^2(N'-N)\\ \widehat{\theta\theta} &= (N+N')(A+\tfrac{1}{2}Mz^2)+B'r^2(N+3N')+B/r^2(N-N')\end{aligned}\right\}\qquad(38)$$

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NOTE ON THE TORSION OF CURVED RODS COMPOSED OF ANISOTROPIC MATERIAL WITH OVAL SECTIONS

By

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Abstract. Stresses and displacements associated with the twist of curved anisotropic rods having sections of oval type are obtained in this note.

Introduction. The problem of finding stresses in helical springs of isotropic material with rings of oval sections has been solved by Michell (1900) and Shepherd (1950). An attempt has been made by the author in this note to solve the same problem when the beam is composed of cylindrically aeolotropic material.

Nomenclature. a_{pq} = elastic constants. ($a_{pq} = a_{qp}$).

r, θ, z = cylindrical coordinates.

u_r, u_θ, u_z = displacements.

$e_{rr}, e_{\theta\theta}, \ldots e_{\theta z}, \ldots$ = strain components.

$\tau_{rr}, \tau_{\theta\theta}, \ldots \tau_{\theta z}, \ldots$ = stress components.

Solution of the problem. Assuming that all stress components except $\tau_{r\theta}$ and $\tau_{\theta z}$ vanish and $\tau_{r\theta}$ and $\tau_{\theta z}$ are functions of r and z only, we get the equation of equilibrium in the form

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{\partial \tau_{\theta z}}{\partial z} + 2\frac{\tau_{r\theta}}{r} = 0 \tag{1.1}$$

In the case of cylindrical anisotropy the stress strain relations [Love (1952)] are given by

$$\begin{vmatrix} e_{rr} \\ e_{\theta\theta} \\ e_{zz} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{vmatrix} \tau_{rr} \\ \tau_{\theta\theta} \\ \tau_{zz} \end{vmatrix}$$

$$e_{r\theta} = a_{44}\tau_{r\theta}, \quad e_{\theta z} = a_{55}\tau_{\theta z} \quad \text{and} \quad e_{rz} = a_{66}\tau_{rz}.$$

With our previous assumptions regarding stress components we deduce that $e_{r\theta}$ and $e_{\theta z}$ are independent of θ and the remaining strain components vanish.

The compatibility equations stand thus

$$\frac{\partial}{\partial z}\left[\frac{\partial e_{r\theta}}{\partial z}-\frac{\partial e_{\theta z}}{\partial r}+\frac{1}{r}e_{\theta z}\right]=0 \qquad (1.2)$$

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(re_{\theta z})\right]-\frac{\partial}{\partial z}\left(\frac{\partial e_{r\theta}}{\partial r}+\frac{2}{r}e_{r\theta}\right)=0 \qquad (1.3)$$

The equation (1.1) is satisfied if we assume

$$\tau_{r\theta}=\frac{1}{r^2}\frac{\partial\varphi}{\partial z} \text{ and } \tau_{\theta z}=-\frac{1}{r^2}\frac{\partial\varphi}{\partial r}. \qquad (1.4)$$

where φ is a function of r and z only.

Then we have $e_{r\theta}=\frac{a_{44}}{r^2}\frac{\partial\varphi}{\partial z}$ and $e_{\theta z}=-\frac{a_{55}}{r^2}\frac{\partial\varphi}{\partial r}$.

Substituting these values in (1.2) and (1.3) we get

$$\frac{\partial}{\partial r}\left[\frac{\partial^2\varphi}{\partial r^2}-\frac{3}{r}\frac{\partial\varphi}{\partial r}+\frac{a_{44}}{a_{55}}\frac{\partial^2\varphi}{\partial z^2}\right]=0$$

$$\frac{\partial}{\partial z}\left[\frac{\partial^2\varphi}{\partial r^2}-\frac{3}{r}\frac{\partial\varphi}{\partial r}+\frac{a_{44}}{a_{55}}\frac{\partial^2\varphi}{\partial z^2}\right]=0$$

Hence we deduce

$$\frac{\partial^2\varphi}{\partial r^2}-\frac{3}{r}\frac{\partial\varphi}{\partial r}+\frac{a_{44}}{a_{55}}\frac{\partial^2\varphi}{\partial z^2}=\text{constant}. \qquad (1.5)$$

If P be a point on the boundary of a cross-section of the rod and (l, m) are direction cosines of outward drawn normal to the boundary at this point, referred to axes of r and z respectively, then at this point.

$$l\tau_{r\theta}+m\tau_{\theta z}=0 \qquad (1.6)$$

Substituting in (1.6) the values of $\tau_{r\theta}$ and $\tau_{\theta z}$ given in (1.4) we obtain

$$\left(l\frac{\partial}{\partial z}-m\frac{\partial}{\partial r}\right)\varphi=0 \text{ or } \frac{\partial\varphi}{\partial s}=0$$

where ds is an element of arc of the boundary of the cross section. Therefore φ = constant on the boundary. Without any loss of generality we take

$$\varphi=0\ldots \qquad (1.7)$$

on the boundary since the constant term does not affect the stresses.

If we suppose that the boundary of the cross-section of the beam intersects the plane $z=0$ at $r=R\pm a$, we may take in the present case

$$\frac{\varphi}{2F}=\{r^2-(R-a)^2\}\{(R+a)^2-r^2\}\{r^2+K_1\}+K_2z^2+6\,\frac{a_{55}}{a_{44}}\,z^2r^1 \qquad (2.1)$$

where F, K_1, K_2 are constants.

φ as assumed in (2.1) will satisfy differential equation (1.5). Therefore $\varphi = 0$ may be taken to be the boundary of the cross-section. The parameters K_1 and K_2 appearing in the right hand side of (2.1) may be suitably chosen so as to obtain different forms of cross-section given by the equation $\varphi = 0$.

If the cross-section be circular in form then the greatest width occurs at $r = R$, $z = \pm a$. Hence if $\varphi = 0$ represents the boundary of cross section, then

$$\varphi = \frac{\partial \varphi}{\partial r} = 0 \qquad \text{where} \quad r = R \qquad \text{and} \quad z = \pm a. \tag{2.2}$$

From these conditions we get

$$K_1 = \frac{a^2}{2} - 3(1+2n)R^2 \tag{2.3}$$

$$K_2 = \frac{a^4}{2} + 2(4+9n)R^4 - 2(2+3n)R^2a^2 \tag{2.4}$$

where n stands for $\dfrac{a_{55}}{a_{44}}$.

The equation of the boundary then becomes

$$z^2 = \frac{\{r^2-(R-a)^2\}\{(R+a)^2-r^2\}\{-a^2-2r^2+6(1+2n)R^2\}}{12nr^4+a^4+4(4+9n)R^4-4(2+3n)a^2R^2} \tag{2.5}$$

The substitution $r = R+x$ transfers the origin to the centre of the cross-section and the distance ϱ of a point on the boundary from the centre is given by $\varrho^2 = x^2+z^2$.

Putting $r = R+x$ in (2.5) we have $\quad z^2 = \varrho^2 - x^2$

$$= \frac{(2R+x+a)(2R+x-a)(a^2-x^2)\{-a^2-4Rx-2x^2+4(1+3n)R^2\}}{(16+48n)R^4+48nR^3x+72nR^2x^2-4(2+3n)R^2a^2+48nRx^3+12nx^4+a^4}$$

Again $\qquad a^2-\varrho^2 = (a^2-x^2)-(\varrho^2-x^2) = (a^2-x^2)-z^2$

$$= \frac{x^2(a^2-x^2)\{(60n+20)R^2+(48n+12)Rx+(12n+2)x^2-a^2\}}{(16+48n)R^4+48nR^3x+72nR^2x^2-4(2+3n)R^2a^2+48nRx^3+12nx^4+a^4}$$

$$= x^2(a^2-x^2)\left[\frac{5}{4R^2}+\frac{3(1-n)x}{4(1+3n)R^3}+\frac{(9+15n)a^2+(2-78n)x^2}{(16+48n)R^4}-\frac{9n(1-n)}{4(1+3n)^2}\frac{x^2}{R^4}+\ldots\right] \tag{2.6}$$

Equation (2.6) shows that the boundary (2.5) fits the circle $a^2 = x^2+z^2$ at the four points whose x and z coordinates are $(0, -a)$, $(0, +a)$, $(a, 0)$ and $(-a, 0)$. It may also be noted from (2.6) that when R is large in comparison to x the boundary (2.5) is approximately circular in form.

The stresses are given by (1.4)

$$\frac{\tau_{r\theta}}{F} = 24nzr^2 + \frac{2z}{r^2}\{a^4 + (16+36n)R^4 - 4(2+3n)a^2R^2\} \qquad (2.7)$$

$$\frac{\tau_{\theta z}}{F} = -48nz^2r + 12r^3 - 4r\{3a^2 + (10+12n)R^2\} + \frac{4}{r}\{(7+12n)R^4$$

$$+3(1+4n)a^2R^2\} \qquad (2\text{-}8)$$

The resultant of the stresses acting on a cross section of the rod is equal to a force P acting along the axis of the rod.

Therefore $$P = \int\int \tau_{\theta z} dr dz \ldots \qquad (2.9)$$

integration being taken over the entire cross section.

Since the boundary given by equation (2.5) is nearly a circle $(x^2+z^2 = a^2)$, a negligible error occurs if the integral (2.9) is taken over the circle $x^2+z^2 = a^2$.

Proceeding in the this manner we obtain

$$P = \pi F[8(7+12n)R^5 + 4(12n-1)a^2R^3 - 8(4n+1)a^4R - 8(R^2-a^2)^{\frac{1}{2}} \times$$

$$\{(7+12n)R^4 + 3(1+4n)a^2R^2\}]^{-} \text{ (approximately).}$$

Now expanding in powers of $\frac{a}{R}$ we get

$$\frac{P}{\pi F a^5} = 16(1+3n)\frac{R}{a} + \frac{(18+36n)}{2}\frac{a}{R} + \frac{59+156n}{16}\frac{a^2}{R^3} + \qquad (2.9)$$

Thus the constant F is found out if P be known.

At the point $z = 0$, $r = R-a$, $\tau_{\theta r} = 0$ and $\tau_{\theta z}$ is maximum.

Hence we obtain

$$\frac{1}{F}(\tau_{\theta z})_{\max.} = 32(1+3n)aR^2 + 32(2+3n)a^2R + 8(5+12n)a^3\left(1-\frac{a}{R}\right)^{-1} \qquad (2.10)$$

When $n=1$ the material is isotropic and the results (2.6), (2.9) and (2.10) agree with those obtained by Shepherd (1950). Now $e_{r\theta} = a_{44}\tau_{r\theta}$ and $e_{\theta z} = a_{55}\tau_{\theta z}$ and all other strain components vanish. Then from strain displacement relations we obtain

$$\left.\begin{array}{l} \dfrac{\partial u_r}{\partial r} = 0, \quad \dfrac{\partial u_\theta}{\partial \theta} + u_r = 0, \quad \dfrac{\partial u_z}{\partial z} = 0, \quad \dfrac{\partial u_z}{\partial r} + \dfrac{\partial u_r}{\partial z} = 0 \\ \\ \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} + \dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r} = a_{44}\tau_{r\theta} \text{ and } \dfrac{\partial u_\theta}{\partial z} + \dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta} = a_{55}\tau_{\theta z} \end{array}\right\} \qquad (3.1)$$

Equations (3.1) are satisfied if we assume

$$u_r = 0, \quad u_z = A\theta \text{ and } u_\theta = \psi(r,z) \qquad (3.2$$

where A is a constant and ψ is a function to be determined.

Therefore from (2.7), (2.8), (3.1) and (3.2) we have

$$\frac{\partial\psi}{\partial r}-\frac{\psi}{r}=a_{44}F\left[24nzr^2+\frac{2z}{r^2}\{a^4+(16+36n)R^4-4(2+3n)a^2R^2\}\right]$$

$$\frac{\partial\psi}{\partial z}+\frac{A}{r}=a_{55}F\left[-48nz^2r+12r^3-4r\{3a^2+(10+12n)R^2\}+\frac{4}{r}\{(7+12n)R^4+3(1+4n)a^2R^2\}\right]$$

From the above relations we obtain

$$\psi(r,\,z)=F\Big[-16a_{55}nz^3r+12a_{55}r^3z-4a_{55}zr\{3a^2+(10+12n)R^2\}$$

$$-\frac{a_{44}z}{r}\{a^4+(16+36n)R^4-4(2+3n)a^2R^2\}\Big] \qquad (3.3)$$

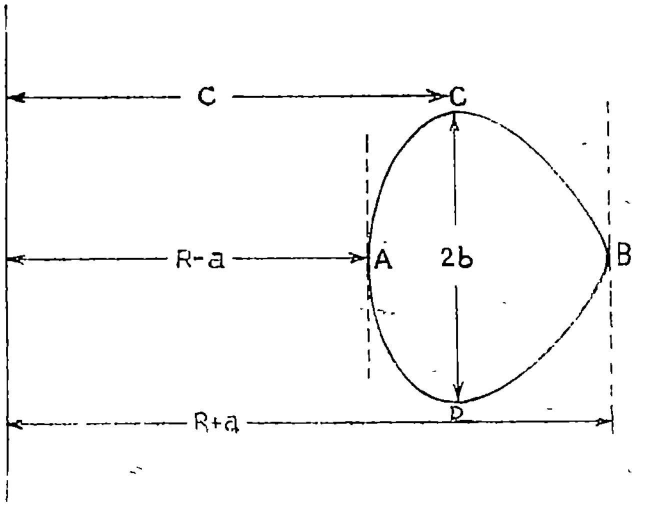
and $A/F=a_{55}[4(7+12n)R^4+12(1+4n)a^2R^2]+a_{44}[a^4+(16+36n)R^4-4(2+3n)a^2R^2]$ (3.4)

Thus displacements are found out from (3.2) – (3.4). For oval sections other than circular in form assuming that the cross-section intersects the plane $z=0$ at $r=R\pm a$. we take as in (2.1)

$$\frac{\varpi}{2F}=\{r^2-(R-a)^2\}\{(R+a)^2-r^2\}\{r^2+K_3\}+K_4z^2+6nz^2r^4 \qquad (4.1)$$

where $$n=\frac{a_{55}}{a_{44}}.$$

Let $\varphi=0$ give equation of the boundary of the cross section. Assuming the maximum width occurs at $r=c$, $z=\pm b(C,\ D)$ (as shown in the figure), we have



$$(i) \quad \varphi = 0 \quad \text{when} \quad r = c, \ z = \pm b \tag{4.2}$$

$$(ii) \quad \frac{\partial \varphi}{\partial r} = 0 \quad \text{when} \quad r = c, \ z = \pm b \tag{4.3}$$

Condition (4 3) gives

$$K_3 = \frac{-5c^4 + 4(R^2 + a^2)c^2 + 12nb^2c^2 - (R^2 - a^2)^2}{2\{c^2 - (R^2 + a^2)\}} \tag{4.4}$$

From condition (4 2) we obtain

$$K_4 = -6nc^4 - \frac{\{c^2 - (R-a)^2\}^2\{(R+a)^2 - c^2\}^2 + 12nb^2c^2\{c^2 - (R-a)^2\}\{(R+a)^2 - c^2\}}{2b^2\{c^2 - (R^2 + a^2)\}} \tag{4.5}$$

Thus the parameters K_3 and K_4 are obtained in terms of b, c, R and a.

In conclusion, I must thank Prof. B. Sen for his kind help in preparation of this note

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NOTE ON FORCED TORSIONAL VIBRATION OF A VISCO-ELASTIC CIRCULAR CYLINDER

By

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(*Received—August 1, 1959*)

Some types of forced torsional vibrations of a circular cylinder of visco-elastic material have been considered by Thompson (1933). We use the stress-strain relations similar to those assumed by him and solve a more complicated type of problem.

Statement. We take the circular cylinder of radius a to be fixed at one end and free at the other which is subjected to an external couple. The material of this cylinder is supposed to satisfy the stress-strain relation

$$\tau_{ij} = \left(\lambda+\lambda'\frac{\partial}{\partial t}\right)e_{kk}\delta_{ij}+\left(\mu+\mu'\frac{\partial}{\partial t}\right)e_{ij}, \tag{1}$$

where τ_{ij} is the stress tensor, e_{ij} is the strain tensor

and
$$\begin{aligned}\delta_{ij} &= 0,\ i\neq j\\ &= 1,\ i=j;\end{aligned}$$

λ, λ', μ, μ' being material constants.

Equation. Choosing cylindrical co-ordinates with its z-axis along the axis of the cylinder and the fixed plane to be the plane $z = 0$, the components of displacement at any point of the cylinder (for pure torsion)

are
$$u_r = 0,\ u_\theta = v(r, z, t)\ u_z = 0, \tag{2}$$
where v is a function of r, z, t. As such the components of strain are

$$e_{rr} = e_{\theta\theta} = e_{zz} = 0,\ e_{r\theta} = \frac{\partial v}{\partial r}-\frac{v}{r},\ e_{\theta z} = \frac{\partial v}{\partial z},\ e_{rz} = 0.$$

Then from (1), the components of stress are of the form

$$\tau_{rr} = \tau_{\theta\theta} = \tau_{zz} = 0,\ \tau_{r\theta} = \left(\mu+\mu'\frac{\partial}{\partial t}\right)e_{r\theta},\ \tau_{\theta z} = \left(\mu+\mu'\frac{\partial}{\partial t}\right)e_{\theta z},\ \tau_{rz} = 0.$$

The equations of motion will then take the form

$$\varrho\frac{\partial^2 v}{\partial t^2} = \left(\mu+\mu'\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}+\frac{\partial^2 v}{\partial z^2}\right), \tag{3}$$

where ϱ is the density of the material.

Assuming $$v = rV(z, t) \tag{4}$$

equation (3) becomes

$$c^2\left(1+k\frac{\partial}{\partial t}\right)\frac{\partial^2 V}{\partial z^2} = \frac{\partial^2 V}{\partial t^2}, \tag{5}$$

where $c^2 = \mu/\varrho$, $k = \mu'/\mu$. For the solution of (5) we take

$$V = T\sin q_n(z), \tag{6}$$

where T is a function of t only, and the expression for V satisfies the condition that $V = 0$ at the fixed end *i.e.* at $z = 0$. So from (5) and (6),

$$\frac{d^2T}{dt^2} + c^2q_n^2k\,\frac{dT}{dt} + c^2q_n^2T = 0. \tag{7}$$

The solution of (7) is $T = C_ne^{s_nt}+D_ne^{s_n't}$, where s_n, s_n' are the roots of the quadratic equation in m, namely

$$m^2 + q_n^2c^2mk + q_n^2c^2 = 0.$$

The particular integral will be of the form

$$v_1 = \Sigma r\sin q_nz(C_ne^{s_nt}+D_ne^{s_n't}). \tag{8}$$

Suppose the couple applied at the free end of the cylinder has its axis parallel to the axis of z and its moment to be decreasing exponentially with time. We are to find the value of v which will satisfy the equation (3) and the following boundary conditions:

$$\left.\begin{aligned}
&\left[\left(\mu+\mu'\frac{\partial}{\partial t}\right)\frac{\partial v}{\partial z}\right]_{z=l} = Qre^{-\alpha t},\ t\geq 0,\ \alpha > 0\\
&[v]_{z=0} = 0,\ t\geq 0\\
&\left[\left(\mu+\mu'\frac{\partial}{\partial t}\right)\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)\right]_{r=a} = 0,\ t\geq 0\\
&v = 0,\ t = 0,\ 0\leq z\leq l\\
&\frac{\partial v}{\partial t} = 0,\ t = 0,\ 0\leq z < l
\end{aligned}\right\} \tag{9}$$

Solution. Let us assume $$v = v_1+v_2, \tag{10}$$

where v_1 stands for the sum of terms as given on the right hand side of (8) and

$$v_2 = Ar\sin pz\,e^{-\alpha t}. \tag{11}$$

The first two conditions of (9) will be satisfied if

$$A = \frac{Q}{p\cos pl(\mu - {}'\alpha)}$$

and $$\cos q_nl = 0 \quad \text{i,e.} \quad q_n = \frac{(2n+1)\pi}{2l}, \qquad n = 0,\ 1,\ 2,\ldots$$

Further the expression for v_2 as given in (11) will satisfy equation (8) if

$$\left(\mu+\mu'\frac{\partial}{\partial t}\right)(-p^2e^{-\alpha t}) = \varrho\alpha^2e^{-\alpha t}$$

or,
$$-p^2(\mu-\mu'\alpha) = \varrho\alpha^2$$

or,
$$p = \frac{\alpha}{c(k\alpha-1)^{\frac{1}{2}}}$$

so p will be real if $k > 1/\alpha$, *i.e.* $\alpha > \mu/\mu'$. The expression for v will then be

$$v = r\sum_{n=0}^{\infty} \sin q_nz(C_ne^{s_nt}+D_ne^{s_n't}) + \frac{Qr\sin pz}{p\cos pl(\mu-\mu'\alpha)}\,e^{-\alpha t}. \tag{12}$$

The third condition of (9) is then identically satisfied and from the last two conditions of (9) we have

$$\sum_{n=0}^{\infty} (C_n+D_n)\sin q_nz = -A\sin pz, \tag{13}$$

$$\sum_{n=0}^{\infty} (s_nC_n+s_n'D_n)\sin q_nz = A\alpha\sin pz. \tag{14}$$

From (13),
$$(C_n+D_n)\int_0^l \sin^2 q_nzdz = -A\int_0^l \sin pz\sin q_nzdz$$

or,
$$C_n+D_n = (-1)^{n+1}\frac{2Ap\cos pl}{l(q_n^2-p^2)} = A.L \text{ (say)}. \tag{15}$$

From (14),
$$s_nC_n+s_n'D_n = -A\alpha L. \tag{16}$$

Solving (15) and (16) for C_n, D_n we have

$$C_n = AL.\frac{s_n'+\alpha}{s_n'-s_n}, \qquad D_n = AL.\frac{s_n+\alpha}{s_n-s_n'}.$$

so v is known completely.

Since $s_n,\ s_n' = -\frac{1}{2}q_n^2c^2k \pm \frac{1}{2}(q_n^4c^4k^2-4q_n^2c^2)^{\frac{1}{2}}$, so if $q_n^2c^2k < 4$,

S_n and $S_n' = -\frac{1}{2}q_n^2c^2k \pm \frac{1}{2}iq_nc(4-q_n^2c^3k^2)^{\frac{1}{2}}$ and they become approximately $-\frac{1}{2}q_n^2c^2k \pm iq_nc$ where $q_n^2c^2k^2 \leqslant 4$. When $q_n^2c^2k^2 \geqslant 4$, s_n and s_n' become approximately $-1/k$ and $-q_n^2e^2k$. The least effective damping factor of the periodic terms under the summation sign in (12) is $e^{-(\pi^2c^2k/8l^2)t}$, and that of the aperiodic terms is $e^{-t/k}$. The part of the motion expressed by v_1 is composed of both periodic and aperiodic parts and this periodic or aperiodic part will survive longest according as $\pi^2c^2k/8l^2 <$ or $> 1/k$. It follows then that the periodic part of the motion survive longest if $\pi^2c^2k/8l^2 < 1/k$, for $1/k < \alpha$, which is the condition for real p. Also the aperiodic part will survive more if $\pi^2c^2k/8l^2 > 1/k$. It can also be seen that in the latter case if $\alpha \geq \pi^2c^2k/8l^2$, then the part of the motion which was expressed by v_2 dies away earlier than the aperiodic part in v_1.

In conclusion, I wish to express my thanks to Prof. Dr. B. B. Sen for his kind help in doing this work.

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TWO THEOREMS ON MEIJER-TRANSFORM

By

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(*Communicated by Dr. N. G. Shabde—Received May 4, 1959.*)

1. **Introduction.** This paper is in continuation of an earlier one (Verma, 1958) by the author in which some integrals involving G-Function have been evaluated by making use of the Laplace and the Meijer Transforms.

Results (3.5) and (4.2) derived here are believed to be new.

Meijer (1941) gave a generalization of the classical Laplace Transform

$$f(p) = p\int_0^\infty e^{-px}h(x)dx,\ R(p) > 0 \tag{1.1}$$

in the form

$$\varphi(p) = p\int_0^\infty e^{-\frac{1}{2}px}(px)^{-k-\frac{1}{2}}W_{k+\frac{1}{2},\,m}(px)h(x)dx, \quad R(p) > 0 \tag{1.2}$$

in which, as is customary, $\varphi(p)$ is said to be the image of the original function $h(x)$. (1.2) reduces to (1.1) when $k = \pm m$ by virtue of the identity

$$W_{\frac{1}{2}\pm m,\,m}(z) \equiv e^{-\frac{1}{2}z}z^{\frac{1}{2}\pm m}$$

We shall denote (1.2) symbolically as

$$\Phi(p) \overset{m}{\underset{m}{=}} h(x)$$

while (1.1) shall be as usual represented by

$$f(p) \doteqdot h(x)$$

2. We start with finding the images of certain functions in the Meijer's transform. These shall be required in our investigations that follow.

If in (1.2) we take

$$h(x) = x^{\rho+k-\frac{1}{2}}e^{\frac{1}{2}ax}W_{\lambda,\mu}(ax)$$

and evaluate the resulting integral [Erdélyi, (1954*b*), p. 411 (45)], we get

$$x^{\rho+k-\frac{1}{2}}e^{\frac{1}{2}ax}W_{\lambda,\mu}(ax)$$

$$\overset{m}{\underset{m}{=}} \frac{a^{-\rho}p^{-k+\frac{1}{2}}}{\Gamma(\frac{1}{2}\pm\mu-\lambda)}G^{2,3}_{3,3}\left(\frac{a}{p}\middle|\begin{matrix}\frac{1}{2}+m,\ \frac{1}{2}-m,\ 1+\lambda+\rho\\ \frac{1}{2}+\mu+\rho,\ \frac{1}{2}-\mu+\rho,\ k+\frac{1}{2}\end{matrix}\right) \qquad (2\ \ )$$

$$R(p) > 0,\ |R(\mu)| + |R(m)| < R(\rho) + 1$$

Similarly (Erdélyi, 1954*b*) p. 411 (46) and p. 412, (54) and (55) yield

$$x^{\rho+k-\frac{1}{2}}e^{-\frac{1}{2}ax}W_{\lambda,\mu}(ax)$$

$$\underset{m}{\overset{m}{=}} a^{-\rho}p^{-k+\frac{1}{2}}G^{2,2}_{3,3}\left(\frac{a}{p}\,\middle|\,\begin{matrix}\frac{1}{2}+m,\ \frac{1}{2}-\mu,\ 1-\lambda+\rho\\ \frac{1}{2}+\mu+\varrho,\ \frac{1}{2}-\mu+\rho,\ k+\frac{1}{2}\end{matrix}\right) \tag{2.2}$$

$$R(p\delta+a)>0,\ |R(\mu)|+|R(m)|<R(\rho)+1$$

$$x^{\rho+k-\frac{1}{2}}e^{-\frac{1}{2}a/x}W_{\lambda,\mu}(a/x)$$

$$\underset{m}{\overset{m}{=}} a^{\rho}p^{-k+\frac{1}{2}}G^{4,0}_{2,4}\left(ap\,\middle|\,\begin{matrix}\frac{1}{2}-k,\ 1-\lambda-\rho\\ \frac{1}{2}+m,\ \frac{1}{2}-m,\ 1+\mu-\rho,\ 1-\mu-\rho\end{matrix}\right) \tag{2.3}$$

$$R(p)>0,\ R(a)\geq 0$$

$$x^{-\rho+k-\frac{1}{2}}e^{\frac{1}{2}a/x}W_{\lambda,\mu}(a/x)$$

$$\underset{m}{\overset{m}{=}} \frac{p^{\rho-k+\frac{1}{2}}}{\Gamma(\frac{1}{2}-\lambda\pm\mu)}G^{4,1}_{2,4}\left(ap\,\middle|\,\begin{matrix}1+\lambda,\ \frac{1}{2}-k-\rho\\ \frac{1}{2}+\mu,\ \frac{1}{2}-\mu,\ \frac{1}{2}+m-\varrho,\ \frac{1}{2}-m-\rho\end{matrix}\right) \tag{2.4}$$

$$|\arg a|<\frac{3\pi}{2},\ R(p)>0,\ R(\lambda+\varrho)<-|R(m)|-\tfrac{1}{2}.$$

Now taking $h(x)\equiv x^{k+l+\delta-\frac{1}{2}}\ {}_2F_1\left[\begin{matrix}a,\ b\\ c\end{matrix}:1-\frac{1}{x}\right]$ in (1.2) and making use of the result (Sharma, 1958, p. 119)

$$\int_0^\infty x^{l+\delta-1}e^{-\frac{1}{2}x}W_{k,m}(x)\ {}_2F_1\left[\begin{matrix}a,\ b\\ c\end{matrix}:1-\frac{1}{x}\right]dx$$

$$=p^{\delta}.\ \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)\Gamma(c-b)\Gamma(b)}\ G^{4,2}_{3,4}\left(p\,\middle|\,\begin{matrix}1-\delta-a,\ 1-\delta-b,\ l-k+1\\ -\delta,\ c-a-b-\delta,\ l+m+\frac{1}{2},\ l-m+\frac{1}{2}\end{matrix}\right)$$

$$R(p)\geq 0,\ R(l\pm m+a+\delta+\tfrac{1}{2})>0,\ R(l\pm m+b+\delta+\tfrac{1}{2})>0,\ R(c-a-b)>0$$

we obtain

$$x^{k+l+\delta-\frac{1}{2}}\ {}_2F_1\left[\begin{matrix}a,\ b\\ c\end{matrix}:1-\frac{1}{x}\right]$$

$$\underset{m}{\overset{m}{=}} \frac{p^{-l-k-\frac{1}{2}}\Gamma(c)}{\Gamma(a)\Gamma(c-a)\Gamma(c-b)\Gamma(b)}\ G^{4,2}_{3,4}\left(p\,\middle|\,\begin{matrix}1-\delta-a,\ 1-\delta-b,\ l-k+\frac{1}{2}\\ -\delta,\ c-a-b-\delta,\ l+m+\frac{1}{2},\ l-m+\frac{1}{2}\end{matrix}\right) \tag{2.5}$$

$$R(p)>0,\ R(l\pm m+a+\delta+\tfrac{1}{2})>0$$

$$R(c-a-b)>0,\ R(l\pm m+b+\delta+\tfrac{1}{2})>0$$

Again using the result (Roopnarain 1957, p. 275)

$$\int_0^\infty t^{\sigma-k-1}e^{-\frac{1}{2}a/t}W_{k,m}\left(\frac{a}{t}\right)G^{l,n}_{p,q}\ \left(\frac{\beta}{t}\,\middle|\,\begin{matrix}\alpha_1\ldots\alpha_p\\ \beta_1\ldots\beta_q\end{matrix}\right)dt$$

$$=2a^{\frac{1}{2}}\int_0^\infty t^{k-\sigma-\frac{1}{2}}k_{2m}2\sqrt{(at)}\,G^{l,n}_{p,q+1}\left(\beta t\,\middle|\,\begin{matrix}\alpha_1\ldots\alpha_p\\ \beta_1\ldots\beta_q,\sigma\end{matrix}\right)dt$$

and evaluating the latter integral (Erdélyi, 1954*b*, p. 421) we obtain

$$x^{-\rho}G^{l,\,n}_{p,\,q}\left(\beta x\,\middle|\,\begin{matrix}\alpha_1\ldots\alpha_p\\ \beta_1\ldots\beta_q\end{matrix}\right)\overset{m}{\underset{m}{=}}p^{\rho}G^{l,\,n+2}_{p+2,\,q+1}\left(\frac{\beta}{p}\,\middle|\,\begin{matrix}\rho+k-m,\ \rho+k+m,\ \alpha_1\ldots\alpha_p\\ \beta_1\ldots\beta_q,\quad \rho+2k\end{matrix}\right) \tag{2.6}$$

$$\begin{matrix}(p+q+1)<2(l+n)\\ |\arg p/\beta|<(l+n-\tfrac{1}{2}p-\tfrac{1}{2}q-\tfrac{1}{2})\pi\end{matrix}\ ;\ R(\rho+k)<1-\tfrac{1}{2}\,|R(2m)|+\min_{1<j<l}R(\beta_j)$$

We shall now proceed to obtain our results.

3. Theorem 1. *If*
$$\varphi(p)\doteqdot h(x)$$
$$p^{2-l}h(p)\overset{m}{\underset{m}{=}}g(x)$$
and
$$p^{-3k+3/2}g(p)\doteqdot f(x),$$
then
$$\varphi(p)=p^{2k-l+3/2}\int_0^\infty G^{2,\,2}_{2,\,3}\left(px\,\middle|\,\begin{matrix}\tfrac{1}{2}+m-k,\ \tfrac{1}{2}-m-k\\ 0,\ l-2k-\tfrac{1}{2},\ \tfrac{1}{2}\end{matrix}\right)f(x)\,dx \tag{3.1}$$
provided the integral is convergent and $R(p)>0$, $R(l-2k)>0$, $R(l-k\pm m)>0$, $R(k)>|R(m)|-\tfrac{1}{2}$.

Proof. Bose (1952, p. 91) has proved the following theorem:

If
$$\psi(p)\doteqdot f(t)$$
$$p^{\nu}e^{-xp^{\mu}}\doteqdot g(x,t)$$
and
$$\Theta(p)\doteqdot F(t)$$
then, provided the integrals converge and the integral
$$\int_0^\infty g(x,v)f(x)\,dx\ \text{is convergent,}$$
$$\int_0^\infty u^{\nu-\mu-1}\psi(u^{\mu})F(u)\,du=\int_0^\infty G(x)f(x)\,dx \tag{A}$$
where
$$G(x)=\int_0^\infty\Theta(v)g(x,v)\frac{dv}{v}.$$

Taking
$$\psi(p)\equiv p^{-2k+3/2}g(p)$$
$$\doteqdot f(t)$$
and $\nu=\tfrac{1}{2}$ and $\mu=-1$ we get $g(x,t)=\pi^{-\frac{1}{2}}t^{-\frac{1}{2}}\cos 2\sqrt{(xt)}$

choosing
$$\Theta(p)\equiv p^{1-\sigma}e^{\frac{1}{2}p/a}W_{k,\,\mu}(p/a)$$
we get (Erdélyi, 1954*a*) p. 294
$$F(t)=\frac{a^{-k}t^{\sigma-k-1}}{\Gamma(\sigma-k)}\,{}_2F_1\left[\begin{matrix}\tfrac{1}{2}-k+\mu,\ \tfrac{1}{2}-k-\mu\\ \sigma-k\end{matrix}\ :-at\right],$$
$$|\arg a|<\pi\quad R(\sigma-k)>0;$$

changing k into $k-l+\frac{1}{2}$ and putting $\sigma=-k+\frac{1}{2}$ and $\mu=m$ we obtain

$$G(x)=\frac{1}{\pi^{\frac{1}{2}}}\int_0^\infty t^{k-1}e^{\frac{1}{2}a/t}W_{k-l+\frac{1}{2},\,m}(t/a)\cos 2\sqrt{(xt)}\,dt.$$

The integral on the right can be evaluated by a result due to Erdélyi (1954*b*, p. 407), so that

$$G(x)=\frac{a^l}{\Gamma(l-k+m)\Gamma(l-k-m)}G^{2,2}_{2,3}\left(ax\,\middle|\,\begin{matrix}\frac{1}{2}+m-k,\ \frac{1}{2}-m-k\\ 0,\ l-2k-\frac{1}{2},\ \frac{1}{2}\end{matrix}\right)\tag{3.2}$$

$$R(x)>0,\quad R(l-2k)>0,\quad R(k)>|R(m)|-\tfrac{1}{2}.$$

Making the desired substutions in (A) and in the left hand side changing u into $1/t$, we get after a little simplification and replacing a by p

$$p\int_0^\infty t^{-l}\,{}_2F_1\left[\begin{matrix}l-k+m,\ l-k-m\\ l-2k\end{matrix}:-\frac{p}{t}\right]g(t)dt$$

$$=\int_0^\infty G^{2,2}_{2,3}\left(px\,\middle|\,\begin{matrix}\frac{1}{2}+m-k,\ \frac{1}{2}-m-k\\ 0,\ l-2k-\frac{1}{2},\ \frac{1}{2}\end{matrix}\right)f(x)dx\tag{3.3}$$

Now Rathie (1953-54), has shown that

If $$\varphi(p)\doteqdot h(x)$$

and $$p^{2-l}h(p)\overset{m}{\underset{m}{=}}g(x)$$

then $$\varphi(p)=\frac{\Gamma(l\pm m-k}{\Gamma(l-2k)}p\int_0^\infty t^{-l}\,{}_2F_1\left[\begin{matrix}l-k+m,\ l-k-m\\ l-2k\end{matrix}:-\frac{p}{t}\right]g(t)dt\tag{B}$$

$R(l-k\pm m)>0$ and provided the integral is convergent.

Since $$p^{-2k+3/2}g(p)\doteqdot f(t)$$

(3.3) with (A) and (B) gives the theorem under conditions stated

Example 1. Taking $f(x)=x^{\nu-\frac{1}{2}}K_{2\mu}(2\sqrt{x})$

$$\doteqdot\frac{\Gamma(\nu\pm\mu+\frac{1}{2})}{2}p^{1-\nu}e^{1/(2p)}W_{-\nu,\,\mu}\left(\frac{1}{p}\right)\equiv p^{-2k+3/2}g(p)$$

$$R(p)>0,\ R(\nu\pm\mu)>-\tfrac{1}{2}.$$

so that $$g(x)=\frac{\Gamma(\nu\pm\mu+\frac{1}{2})}{2}x^{2k-\nu-\frac{1}{2}}e^{1/(2x)}W_{-\nu,\,\mu}\left(\frac{1}{x}\right)$$

the Meijer-transform of which is obtained by (2.4) giving

$$p^{2-l}h(p)=\tfrac{1}{2}p^{\nu-2k+\frac{1}{2}}G^{4,1}_{2,1}\left(p\,\middle|\,\begin{matrix}1-\nu,\ \frac{1}{2}-\nu\\ \frac{1}{2}+\mu,\ \frac{1}{2}-\mu,\ \frac{1}{2}+k+m-\nu,\ \frac{1}{2}+k-m-\nu\end{matrix}\right)$$

$$R(k\pm m)>\tfrac{1}{2}$$

and $$h(x) = \frac{x^{l+\nu-2k-3/2}}{2} G^{4,1}_{2,4}\left(x \middle| \begin{matrix} 1-\nu,\ \frac12-\nu \\ \frac12+\mu,\ \frac12-\mu,\ \frac12+k+m-\nu,\ \frac12-k-m-\nu \end{matrix}\right)$$

The Laplace Transform of this is given by [Erdélyi, (1954*a*), p. 222],

$$\varphi(p) = \frac{p^{2k-l-\nu+3/2}}{2} G^{4,2}_{3,4}\left(\frac{1}{p} \middle| \begin{matrix} 2k-l-\nu+3/2,\ 1-\nu,\ \frac12-\nu \\ \frac12+\mu,\ \frac12-\mu,\ \frac12+k+m-\nu,\ \frac12+k-m-\nu \end{matrix}\right)$$

$$|\arg p| < 2\pi, \quad R(l-k\pm m) > 0, \quad R(l-2k+\nu\pm\mu) > 0$$

Hence putting $\frac12+m-k = a_1$, $\frac12-m-k = a_2$ and $l-2k-\frac12 = b$

we get finally $$\int_0^\infty x^{\nu-\frac12} G^{2,2}_{2,3}\left(px \middle| \begin{matrix} a_1,\ a_2 \\ 0,\ b,\ \frac12 \end{matrix}\right) K_{2\mu}(2x^{\frac12})\,dx$$

$$= \tfrac12 p^{-\nu} G^{4,2}_{3,4}\left(\frac{1}{p} \middle| \begin{matrix} 1-\nu-b,\ 1-\nu,\ \frac12-\nu \\ \frac12+\mu,\ \frac12-\mu,\ 1-a_2,\ 1-a_1 \end{matrix}\right) \tag{3.4}$$

$$R(p) > 0,\ R(b+\tfrac12) > 0,\ R(1-a_1-b) > 0,\ R(1-a_2-b) > 0$$

$$R(1-a_1) > 0,\ R(1-a_2) > 0,\ R(b+\nu\pm\mu+\tfrac12) > 0$$

which is a particular case of a known result, (Erdélyi, 1954*b*, p. 421)

Example 2. Taking $$f(x) = 2^{-2k+\frac12} x^{-k+\frac14} P^{2k-\frac12}_{2\mu-\frac12}(1+x)^{\frac12}$$

$$\doteqdot p^k e^{\frac12 p} W_{k,\mu}(p) \equiv p^{-2k+3/2} g(p), \quad k < 3/4$$

$P^\mu_\nu(z)$ being the Legendre's Associated function of the first kind, we get

$$g(x) = x^{3k-3/2} e^{\frac12 x} W_{k,\mu}(x)$$

The Meijer-Transform of this by (2.1) is

$$p^{s-l} h(p) = \frac{p^{-k+\frac14}}{\Gamma(\frac12-k\pm\mu)} G^{2,3}_{3,3}\left(\frac{1}{p} \middle| \begin{matrix} \frac12+m,\ \frac12-m,\ 3k \\ 2k+\mu-\frac12,\ 2k-\mu-\frac12,\ k+\frac12 \end{matrix}\right)$$

$$|R(\mu)| + |R(m)| < R(2k)$$

so that $$h(x) = x^{l-k-3/2} G^{2,3}_{3,3}\left(\frac{1}{x} \middle| \begin{matrix} \frac12+m,\ \frac12-m,\ 3k \\ 2k+\mu-\frac12,\ 2k-\mu-\frac12,\ k+\frac12 \end{matrix}\right)$$

Now using (Erdélyi, 1953, p. 209)

$$G^{m,n}_{p,q}\left(\frac{1}{x} \middle| \begin{matrix} a_r \\ b_s \end{matrix}\right) = G^{n,m}_{q,p}\left(x \middle| \begin{matrix} 1-b_s \\ 1-a_r \end{matrix}\right) \tag{C}$$

and finding its Laplace Transform (Erdélyi, 1954*a*, p. 222)

we obtain $$\varphi(p) = p^{k-l+3/2} G^{3,3}_{4,3}\left(\frac{1}{p} \middle| \begin{matrix} k-l+3/2,\ 3/2-\mu-2k,\ 3/2+\mu-2k,\ \frac12-k \\ \frac12-m,\ \frac12+m,\ 1-3k \end{matrix}\right)$$

$$|\arg p| < 2\pi,\ R(p) > 1,\ R(l\pm m-k) > 0,\ R(l-4k+\tfrac12) > 0$$

Hence
$$\int_0^\infty x^{-k+\frac{1}{2}} G^{2,2}_{2,3}\left(px\,\middle|\,\begin{matrix}\frac{1}{2}+m-k,\ \frac{1}{2}-m-k\\ 0,\ l-2k-\frac{1}{2},\ \frac{1}{2}\end{matrix}\right) P^{2k-\frac{1}{2}}_{2\mu-\frac{1}{2}}\,P(1+x)^{\frac{1}{2}}dx$$

$$= 2^{2k-\frac{1}{2}}p^{-k}G^{3,3}_{4,3}\left(\frac{1}{p}\,\middle|\,\begin{matrix}3/2+k-l,\ 3/2+\mu-2k,\ 3/2+\mu-2k,\ \frac{1}{2}-k\\ \frac{1}{2}-m,\ \frac{1}{2}+m,\ 1-3k\end{matrix}\right) \tag{3.5}$$

$$R(p)>1,\ R(l-2k)>0,\ R(l-k\pm m)>0,\ k<\tfrac{3}{4}$$
$$R(l-4k+\tfrac{1}{2})>0,\ |R(\mu)|+|R(m)|<R(2k);\ R(k)>|R(m)|-\tfrac{1}{2}.$$

4. Theorem 2. *If*
$$\varphi(p)\ \overset{m}{\underset{m}{=}}\ h(x)$$
$$p^{2-l}h(p)\doteqdot g(x)$$
and
$$p^{l-2k-\frac{1}{2}}g(1/p)\doteqdot f(x),$$
then
$$\varphi(p) = p^{-2k+\frac{1}{2}}\int_0^\infty G^{2,2}_{2,3}\left(\frac{x}{p}\,\middle|\,\begin{matrix}\frac{1}{2}+m-k,\ \frac{1}{2}-m-k\\ 0,\ l-2k-\frac{1}{2},\ \frac{1}{2}\end{matrix}\right)f(x)dx \tag{4.1}$$
provided integral is convergent and $R(p)>0$ $R(l-2k)>0$, $R(l-k\pm m)>0$, $R(k)>|R(m)|-\frac{1}{2}$.

Proof. In 3(A) taking $\psi(p)\equiv p^{l-3k-\frac{1}{2}}g(1/p)$ and proceeding exactly in the same way as in theorem 1, we get the theorem on using the following one due to Rathie (1953-54, p. 6)

If
$$\varphi(p)\ \overset{m}{\underset{m}{=}}\ h(x)$$
and
$$p^{2-l}h(p)\doteqdot g(x)$$
then
$$\varphi(p) = \frac{\Gamma(l\pm m-k)}{\Gamma(l-2k)}p^{1-l}\int_0^\infty {}_2F_1\left[\begin{matrix}l-k+m,\ l-k-m\\ l-2k\end{matrix}\ :-\frac{t}{p}\right]g(t)dt \tag{D}$$
provided the integral is convergent and $R(l-k\pm m)>0$

Example 1. If we take

$$f(x) = x^{\sigma-1}\left[\frac{\Gamma(-2\mu)x^{\mu+\frac{1}{2}}}{\Gamma(\frac{1}{2}-k-\mu)\,\Gamma(\frac{1}{2}+\mu+\sigma)}\,{}_1F_2\left(\begin{matrix}\frac{1}{2}+\mu-k\\ 1+2\mu,\ 1+\mu-\sigma\end{matrix}\ :x\right)\right.$$
$$\left.+\frac{\Gamma(2\mu)x^{-\mu+\frac{1}{2}}}{\Gamma(\frac{1}{2}-k+\mu)\Gamma(\frac{1}{2}-\mu+\sigma)}\,{}_1F_2\left(\begin{matrix}\frac{1}{2}-\mu+k\\ 1-2\mu,\ 1-\mu-\sigma\end{matrix}\ :x\right)\right]$$
$$\doteqdot p^{-\sigma+1}e^{\frac{1}{2}p}W_{k,\mu}(1/p)\equiv p^{l-2k-\frac{1}{2}}g(1/p),$$
$$R(\tfrac{1}{2}\pm\mu+\sigma)>0$$

so that
$$g(x) = x^{l-2k+\sigma-3/2}e^{\frac{1}{2}x}W_{k,\mu}(x)$$
$$\doteqdot\frac{\Gamma(\mu+l-2k+\sigma)\Gamma(l-2k-\mu+\sigma)}{\Gamma(l-3k+\sigma+\frac{1}{2})}p^{2k-l-\sigma-\mu+1}$$
$$\times{}_2F_1\left[\begin{matrix}\mu+l+\sigma-2k,\ \mu-k+\frac{1}{2}\\ l-3k+\sigma+\frac{1}{2}\end{matrix}\ :1-\frac{1}{p}\right]\equiv p^{2-l}h(p)$$

Then on finding $h(x)$, its Meijer Transform is given by (2.5)

$$\varphi(p)=\frac{p^{-l-k+\frac{1}{2}}\Gamma(l-3k+\sigma+\frac{1}{2})}{\Gamma(\frac{1}{2}-k+\mu)\Gamma(\frac{1}{2}-k-\mu)\Gamma(l-2k+\sigma\pm\mu)}$$

$$\times G^{4,\,2}_{3,\,4}\left(p\,\middle|\,\begin{matrix}3/2+k,\ 1+\sigma+l,\ l-k+\frac{1}{2}\\ l-k+\sigma+\mu+\frac{1}{2},\ l-k-\mu+\sigma+\frac{1}{2},\ l+m+\frac{1}{2},\ l-m+\frac{1}{2}\end{matrix}\right)$$

$$R(l+m-k)>0,\ R(\mu)<0$$

Hence we get

$$\int_0^\infty x^{\sigma-1}G^{2,\,2}_{2,\,3}\left(\frac{x}{p}\,\middle|\,\begin{matrix}\frac{1}{2}+m-k,\ \frac{1}{2}-m-k\\ 0,\ l-2k-\frac{1}{2},\ \frac{1}{2}\end{matrix}\right)\left[\sum_{\mu_1-\mu}{}^{*}\left\{\frac{\Gamma(-2\mu)x^{\mu+\frac{1}{2}}\ {}_1F_2\left({\frac{1}{2}-k+\mu \atop 1+2\mu,\ 1+\mu+\sigma}:x\right)}{\Gamma(\frac{1}{2}-k-\mu)\Gamma(\frac{1}{2}+\mu+\sigma)}\right\}\right]dx$$

$$=\frac{p^{-l+k}}{\Gamma(\frac{1}{2}-k\pm\mu)}\,G^{4,\,2}_{3,\,4}\left(p\,\middle|\,\begin{matrix}\frac{3}{2}+k,\ 1+\sigma+l,\ l-k+\frac{1}{2}\\ l-k+\sigma+\mu+\frac{1}{2},\ l-k-\mu+\sigma-\frac{1}{2},\ l+m+\frac{1}{2},\ l-m+\frac{1}{2}\end{matrix}\right)\qquad(4.2)$$

$$\mathrm{R}(p)>0,\ R(l-2k)>0,\ R(l\pm m-k)>0,\ R(\tfrac{1}{2}\pm\mu+\sigma)>0$$

$$R(l-2k+\sigma\pm\mu)>0,\ R(\tfrac{1}{2}-\sigma\pm m)>0,\ R(\mu)<0$$

Example 2. Next taking

$$f(x)=x^{-\sigma}G^{n,\,r}_{h,\,s}\left(x\,\middle|\,\begin{matrix}\alpha_1\ldots\alpha_h\\ \beta_1\ldots\beta_s\end{matrix}\right)\doteqdot p^{\sigma}G^{n,\,r+1}_{h+1,\,s}\left(\frac{1}{p}\,\middle|\,\begin{matrix}\sigma,\ \alpha_1\ldots\alpha_h\\ \beta_1\ldots\ldots\beta_s\end{matrix}\right)\equiv p^{l-2k-\frac{1}{2}}g\left(\frac{1}{p}\right)$$

$$\begin{matrix}(h+s)<2(n+r)\\ R(\sigma-1)>R(\beta_j);\ j=1\ldots n\end{matrix}\ ;\ |\arg p|<(n+r-\tfrac{1}{2}h-\tfrac{1}{2}s)\pi$$

so that

$$g(x)=x^{l-2k-\sigma-\frac{1}{2}}G^{n,\,r+1}_{h+1,\,s}\left(x\,\middle|\,\begin{matrix}\sigma,\ \alpha_1\ldots\alpha_h\\ \beta_1\ldots\ldots\beta_s\end{matrix}\right)$$

$$\doteqdot p^{\sigma+2k-l+\frac{1}{2}}G^{n,\,r+1}_{h+2,\,s}\left(\frac{1}{p}\,\middle|\,\begin{matrix}2k+\sigma-l+\frac{1}{2},\ \sigma,\ \alpha_1\ldots\alpha_h\\ \beta_1\ldots\ldots\beta_s\end{matrix}\right)\equiv p^{2-l}h(p)$$

$$R(\sigma+2k-l-\tfrac{1}{2})>R(\beta_j);\ j=1\ldots n.$$

Finding $h(x)$ and using 3(C) in order to find its Meijer-Transform, we get by virtue of (2.6)

$$\varphi(p)=p^{3/2-\sigma-2k}G^{r+2,\,n+2}_{s+2,\,h+3}\left(\frac{1}{p}\,\middle|\,\begin{matrix}\frac{3}{2}-k-\sigma-m,\ \frac{3}{2}-k-\sigma+m,\ 1-\beta_1,\ \ldots1-\beta_s\\ \frac{1}{2}-2k-\sigma+l,\ 1-\sigma,\ 1-\alpha_1,\ \ldots1-\alpha_h,\ \frac{3}{2}-\sigma\end{matrix}\right)$$

$$R(\tfrac{1}{2}-\sigma-k)<-\tfrac{1}{2}|R(2m)|+\min_{i<j<r}(1-\alpha_j)$$

---

* $\sum_{\mu_1-\mu}$ means that to the expression following it a similar expression with $(-\mu)$ written in place of μ is to be added.

Hence putting $\frac{1}{2}+m-k = a_1,\ \frac{1}{2}-m-k = a_2$ and $l-2k-\frac{1}{2} = b$

we get
$$\int_0^\infty x^{-\sigma} G^{2,2}_{2,3}\left(\frac{x}{p}\middle|\begin{matrix}a_1, & a_2\\ 0, & b, & \frac{1}{2}\end{matrix}\right) G^{n,r}_{h,s}\left(x\middle|\begin{matrix}\alpha_1\ldots\alpha_h\\ \beta_1\ldots\beta_s\end{matrix}\right)dx$$

$$= p^{1-\sigma} G^{r+2,\,n+2}_{s+2,\,n+3}\left(\frac{1}{p}\middle|\begin{matrix}1+a_2-\sigma,\ 1+a_1-\sigma,\ 1-\beta_1,\ \ldots 1-\beta_s\\ b-\sigma,\ 1-\sigma,\ 1-\alpha_1\ldots 1-\alpha_h,\ \frac{3}{2}-\sigma\end{matrix}\right) \tag{4.3}$$

$$R(p) > 0,\ R(b+\tfrac{1}{2}) > 0,\ R(1-a_1) > 0,\ R(1-a_2) > 0$$

$$R(1-a_1+b) > 0,\ R(1-a_2+b) > 0,\ R(\sigma+b) > R(\beta_j),\ j = 1\ldots n$$

$$|\arg(p)| < (n+r-\tfrac{1}{2}h-\tfrac{1}{2}s)\pi,\ R[\tfrac{1}{2}(a_1+a_2)-\sigma] < -\tfrac{1}{2}|R(a_1-a_2)| + \min_{1<j<r}(1-\alpha_j)$$

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ABEL SUMMABILITY OF THE DOUBLE SERIES SUCCESSIVELY DERIVED FROM THE CONJUGATE OF THE DOUBLE FOURIER SERIES

By

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The purpose of this paper is to generalize a result of Misra (1947) to double series. Let the Abel sum of the conjugate of the double Fourier series associated with a function $f(x, y)$ which is integrable (L) in the cell $Q(-\pi, -\pi; \pi, \pi)$ and defined outside of Q by periodicity be denoted by $A(x, y; r, p)$.

Then if $K(r, s) = \dfrac{r \sin s}{1-2r\cos s+r^2}$ is Poisson's conjugate kernel, we have

$$A(x, y; r, p) = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u, v)K(r, u-x)K(p, v-y)\,du\,dv.$$

Notation. Let
$$\begin{aligned}\varphi_{jk}(x, y; \gamma, \varrho) = {} & f(x+\gamma, y+\varrho)-f(x+\gamma, y)-f(x, y+\varrho)+f(x, y)\\ & -(-1)^j[f(x-\gamma, y+\varrho)-f(x-\gamma, y)-f(x, y+\varrho)+f(x, y)]\\ & -(-1)^k[f(x+\gamma, y-\varrho)-f(x+\gamma, y)-f(x, y-\varrho)+f(x, y)]\\ & +(-1)^{j+k}[f(x-\gamma, y-\varrho)-f(x-\gamma, y)-f(x, y-\varrho)+f(x, y)],\end{aligned}$$

and $\ddot{\varphi}_{jk}(x, y; s, t) = \displaystyle\int_0^t\int_0^s \varphi_{jk}(x, y; u, v)\,du\,dv$ where j, k are positive integers or zero.

Theorem. If $\ddot{\varphi}_{jk}(x, y; s, t) = o(s^{j+1}t^{k+1})$ as $(s, t)\to 0$, $o(s^{j+1})$ as $s\to 0$ uniformly on $0 < t < \pi$, $o(t^{k+1})$ as $t\to 0$ uniformly on $0 < s < \pi$, then

$$\lim_{(\alpha, \beta)\to 0}\left\{\frac{\partial^{j+k}A(x, y; r, p)}{\partial x^j\partial y^k}-\frac{(-1)^{j+k}}{4\pi^2}\int_{\beta}^{\pi}\int_{\alpha}^{\pi}\varphi_{jk}(x, y; \gamma, \varrho)\,\frac{d^j\cot\frac{1}{2}\gamma}{d\gamma^j}\;\frac{d^k\cot\frac{1}{2}\varrho}{d\varrho^k}\,d\gamma\,d\varrho\right\} = 0$$

where $\alpha = \arcsin(1-r)$, $\beta = \arcsin(1-p)$.

We shall require several lemmas.

Lemma 1. *If* $W(r, \gamma) = \dfrac{2r-(1+r^2)\cos\gamma}{1+r^2-2r\cos\gamma}$, $\nabla(r, \gamma) = \dfrac{1+r^2-2rW}{(1+r)^2(1-r)^2}$, $K(r, \gamma) = r\sin\gamma\nabla$,

then $\dfrac{d^{2n}K}{d\gamma^{2n}} = r\sin\gamma\nabla^{n+1}P_n$, $\dfrac{d^{2n+1}K}{d\gamma^{2n+1}} = r\nabla^{n+1}P_{n+1}$ for $n = 0, 1, 2, \ldots$, *where* P_n, P_{n+1} *are polynomials in* r, W, $\cos\gamma$, *with* P_n *of degree* n *and* P_{n+1} *of degree* $n+1$ *in* W.

Proof of Lemma 1. The result follows by mathematical induction.

Lemma 2. $$\int_{-\pi}^{\pi} \frac{d^nK}{d\gamma^n}\, d\gamma = 0 \text{ for } n = 0, 1, 2, \ldots.$$

Proof of Lemma 2. This is Lemma 6 of Misra, (1947) and follows from Lemma 1.

Lemma 3. *If* $q(r, \gamma) = \dfrac{(1-r)^2 \cot \frac{1}{2}\gamma}{2(1+r^2-2r\cos\gamma)}$, *then* $\dfrac{d^nq(r,\gamma)}{d\gamma^n} = (1-r)^2 O\left(\dfrac{1}{\gamma^{n+3}}\right)$ *as* $\gamma = \textit{arc sin}\,(1-r) \to 0$, $(1-r)^2 O(1)$ *for* $\gamma > \xi > 0$ *as* $r \to 1$ *with* $n = 0, 1, 2, \ldots.$

Proof of Lemma 3. This is Lemma 5 of Misra (1947).

Lemma 4. *For j, k positive integers or zero,*

$$\frac{\partial^jK(r,\gamma)}{\partial\gamma^j}\,\frac{\partial^kK(p,\varrho)}{\partial\varrho^k} = \tfrac{1}{4}\left\{\frac{d^j(\cot\frac{1}{2}\gamma)}{d\gamma^j}\,\frac{d^k(\cot\frac{1}{2}\varrho)}{d\varrho^k}\right\}$$
$$-\left\{\frac{\partial^jq(r,\gamma)}{\partial\gamma^j}\,\frac{\partial^kq(p,\varrho)}{\partial\varrho^k} + \frac{\partial^jq(r,\gamma)}{\partial\gamma^j}\,\frac{\partial^kK(p,\varrho)}{\partial\varrho^k} + \frac{\partial^jK(r,\gamma)}{\partial\gamma^j}\,\frac{\partial^kq(p,\varrho)}{\partial\varrho^k}\right\}$$

Proof of Lemma 4. This follows from the definition of the functions concerned.

Lemma 5. *If $\ddot{\varphi}_{jk}(x, y\,;\gamma, \varrho)$ satisfies the hypothesis of the theorem, then*

$$(5.11) = \int_0^\beta \int_0^\alpha \varphi_{jk} \frac{\partial^jK}{\partial\gamma^j}\,\frac{\partial^kK}{\partial\varrho^k}\, d\gamma d\varrho = o(1),\ (5.12) = \int_0^\beta \int_\alpha^\pi \varphi_{jk} \frac{\partial^jK}{\partial\gamma^j}\,\frac{\partial\varrho K}{\partial\varrho^k}\, d\gamma d\varrho = o(1),$$

$$(5.13) = \int_\beta^\pi \int_0^\alpha \varphi_{jk} \frac{\partial^jK}{\partial\gamma^k}\,\frac{\partial^kK}{\partial\varrho^k}\, d\gamma d\varrho = o(1)\,;\ (5.14) = \int_\beta^\pi \int_\alpha^\pi \varphi_{jk} \frac{\partial^jq}{\partial\gamma^j}\,\frac{\partial^kq}{\partial\varrho^k}\, d\gamma d\varrho = o(1),$$

$$(5.15) = \int_\beta^\pi \int_\alpha^\pi \varphi_{jk} \frac{\partial^jK}{\varrho\gamma^j}\,\frac{\partial^kK}{\partial\varrho^k}\, d\gamma d\varrho = o(1),\ (5.16) = \int_\beta^\pi \int_\alpha^\pi \varphi_{jk} \frac{\partial^jq}{\partial\gamma^j}\,\frac{\partial^kK}{\partial\varrho^k}\, d\gamma d\varrho = o(1),$$

as $[\alpha = \textit{arc sin}\,(1-r),\ \beta = \textit{arc sin}\,(1-p)] \to 0$

Proof of Lemma 5. Let $\dfrac{\partial^jH}{\partial\gamma^j}$ denote either $\dfrac{\partial^jK}{\partial\gamma^j}$ or $\dfrac{\partial^jq}{\partial\gamma^j}$. We take general limits over the rectangle $(a, b\,; c, d)$ where $0 < a < c < \pi$, $0 < b < d < \pi$ and integrate

$(5.20) = \displaystyle\int_b^d \int_a^c \ddot{\varphi}_{jk}(x, y\,;\gamma, \varrho)\,\frac{\partial^jH(r,\gamma)}{\partial\gamma^j}\,\frac{\partial^kH(p,\varrho)}{\partial\varrho^j}\, d\gamma d\varrho$ by parts to obtain sum the of the following: $(5.21) = \dfrac{\partial^jH(r,c)}{\partial\gamma^j}\,\dfrac{\partial^kH(p,d)}{\partial\varrho^k}\,\ddot{\varphi}_{jk}(x, y\,; c, d)$, $(5.22) = \dfrac{-\partial^jH(r,c)}{\partial\gamma^j}\,\dfrac{\partial^kH(p,b)}{\partial\varrho^k}\,\ddot{\varphi}_{jk}(x,y;c,b)$

$(5.23) = \dfrac{\partial^jH(r,a)}{\partial\gamma^j}\,\dfrac{\partial^kH(p,b)}{\partial\varrho^k}\,\ddot{\varphi}_{jk}(x, y\,; a, b)$, $(5.24) = \dfrac{-\partial^jH(r,a)}{\partial\gamma^j}\,\dfrac{\partial^kH(p,d)}{\partial\varrho^k}\,\ddot{\varphi}(x, y\,; a, d)$,

$$(5.25) = \frac{\partial^j H(r, a)}{\partial \gamma^j} \int_b^d \ddot{\varphi}_{jk}(x, y\,; a, \varrho)\, \frac{\partial^{k+1} H(p, \varrho)}{\partial \varrho^{k+1}}\, d\varrho,$$

$$(5.26) = \frac{-\partial^j H(r, c)}{\partial \gamma^j} \int_b^d \ddot{\varphi}_{jk}(x, y\,; c.\, \varrho) \frac{\partial^{k+1} H(p, \varrho)}{\partial \varrho^{k+1}}\, d\varrho,$$

$$(5.27) = \frac{\partial^k H(p, b)}{\partial \varrho^k} \int_a^c \ddot{\varphi}_{jk}(x, y\,; \gamma, b)\, \frac{\partial^{j+1} H(r, \gamma)}{\partial \gamma^{j+1}}\, d\gamma,$$

$$(5.28) = \frac{-\partial^k H(p, d)}{\partial \varrho^k} \int_a^c \ddot{\varphi}_{jk}(x, y, \gamma, d)\, \frac{\partial^{j+1} H(r, \gamma)}{\partial \gamma^{j+1}}\, d\gamma,$$

$$(5.29) = \int_b^d \int_a^c \dot{\varphi}_{jk}(x, y\,; \gamma, \varrho)\, \frac{\partial^{j+1} H(r, \gamma)}{\partial \gamma^{j+1}}\, \frac{\partial^{k+1} H(p, \varrho)}{\partial \varrho^{k+1}}\, d\gamma d\varrho.$$

In consideration of (5.11), let $\frac{\partial^j H}{\partial \gamma^j} = \frac{\partial^j K}{\partial \gamma^j}$, $\frac{\partial^k H}{\partial \varrho^k} = \frac{\partial^k K}{\partial \varrho^k}$; then if we let $a \to 0$, $b \to 0$, $c = \alpha = \text{arc sin } (1-r) \neq 0$, we have

$$(5.22) = C(1) O(1) o(b^{k+1}) = o(1),$$

$$(5.23) = O(1) O(1) o(a^{j+1} b^{k+1}) = o(1), \quad (5.24) = O(1) O(1) o(a^{j+1}) = o(1),$$

$$(5.25) = O(1) o\left(a^{j+1} \int_b^\beta d\varrho\right) o(1), \; (5.27) = O(1) o\left(b^{k+1} \int_a^\alpha d\gamma\right) = o(1).$$

If we now let $[\alpha = \text{arc sin } (1--r),\ \beta = \text{arc sin } (1-p)] \to 0$,

we have $$(5.21) = O\left(\frac{1}{\alpha^{j+1}}\right) O\left(\frac{1}{\beta^{k+1}}\right) o(\alpha^{j+1} \beta^{k+1}) = o(1)$$

$$(5.23) = O\left(\frac{1}{\alpha^{j+1}}\right) o\left(\beta^{j+1} \int^\beta \varrho^{k+1} \begin{bmatrix} \dfrac{\sin \varrho}{(1-p)^{k+3}} \\ \dfrac{1}{(1-p)^{k+2}} \end{bmatrix} d\varrho \right) = o(1),$$

$$(5.28) = O\left(\frac{1}{\beta^{k+1}}\right) o\left(\beta^{k+1} \int_0^\alpha \gamma^{j+1} \begin{bmatrix} \dfrac{\sin \gamma}{(1-r)^{j+3}} \\ \dfrac{1}{(1-r)^{j+2}} \end{bmatrix} d\gamma \right) = o(1),$$

$$(5.29) = \left(\int_0^\beta \int_0^\alpha \gamma^{j+1} \varrho^{k+1} \begin{bmatrix} \dfrac{\sin \gamma}{(1-r)^{j+3}} \\ \dfrac{1}{(1-r)^{j+2}} \end{bmatrix} \begin{bmatrix} \dfrac{\sin \varrho}{(1-p)^{k+3}} \\ \dfrac{1}{(1-p)^{k+2}} \end{bmatrix} d\gamma d\varrho \right) = o(1),$$

by Lemma 1, whence (5.11) $= o(1)$,

In consideration of (5.12), let $\frac{\partial^j H}{\partial \gamma^j} = \frac{\partial^j K}{\partial \gamma^j}$, $\frac{\partial^k H}{\partial \varrho^k} = \frac{\partial^k K}{\partial \varrho^k}$; thet if we let $b \to 0$, $c = \pi$, $a = \alpha = \text{arc sin }(1-r) \neq 0$, $d = \beta = \text{arc sin }(1-p) \neq 0$, we have $(5.22) = O(1)O(1)o(b^{k+1}) = o(1)$,

$(5.23) = O(1)O(1)o(b^{k+1}) = o(1)$, $(5.27) = O(1)O\left((b^{k+1}\int_\alpha^\pi d\gamma\right) = o(1)$. If we now let

$[\alpha = \text{arc sin }(1-r),\ \beta = \text{arc sin }(1-\varrho)] \to 0$ we have $(5.21) = O(1)O\left(\frac{1}{\beta^{k+1}}\right)o(\beta^{k+1}) = o(1)$,

$$(5.24) = O\left(\frac{1}{\alpha^{j+1}}\right)O\left(\frac{1}{\beta^{k+1}}\right)o(\alpha^{j+1}\beta^{k+1}) = o(1),$$

$$(5.25) = O\left(\frac{1}{\alpha^{j+1}}\right)o\left(\alpha^{j+1}\int_0^\beta \varrho^{k+1}\left|\begin{bmatrix}\dfrac{\sin\varrho}{(1-p)^{k+3}}\\[2ex] \dfrac{1}{(1-p)^{k+2}}\end{bmatrix}\right| d\varrho\right) = o(1),$$

$$(5.26) = O(1)o\left(\int_0^\beta \varrho^{k+1}\left|\begin{bmatrix}\dfrac{\sin\varrho}{(1-p)^{k+3}}\\[2ex] \dfrac{1}{(1-p)^{k+2}}\end{bmatrix}\right| d\varrho\right) = o(1),$$

$$(5.28) = -\frac{\partial^k K(p,\beta)}{\partial\varrho^k}\left(\int_\alpha^\xi + \int_\xi^\pi\right)\ddot{\varphi}_{jk}(x, y;\gamma,\beta)\frac{\partial^{j+1}K(r,\gamma)}{\partial\gamma^{j+1}}\,d\gamma$$

$$= O\left(\frac{1}{\beta^{k+1}}\right)o\left(\beta^{k+1}\int_\alpha^\xi \gamma^{j+1}\left|\begin{bmatrix}\dfrac{\sin\gamma}{(1-r)^{j+3}}\\[2ex] \dfrac{1}{(1-r)^{j+2}}\end{bmatrix}\right| d\gamma\right) + O\left(\frac{1}{\beta^{k+1}}\right)o\left(\beta^{k+1}\int_\xi^\pi d\gamma\right) = o(1) + o(1) = o(1).$$

$$(5.29) = o\left(\int_0^\beta\int_\alpha^\pi \gamma^{j+1}\varrho^{k+1}\left|\begin{bmatrix}\dfrac{\sin\gamma}{(1-r)^{j+3}}\\[2ex] \dfrac{1}{(1-r)^{j+2}}\end{bmatrix}\right|\left|\begin{bmatrix}\dfrac{\sin\varrho}{(1-p)^{k+3}}\\[2ex] \dfrac{1}{(1-p)^{k+2}}\end{bmatrix}\right| d\gamma\, d\varrho\right) = o(1) \text{ by Lemma 1, whence}$$

$(5.12) = o(1)$.

$(5.13) = o(1)$ by analogy with (5.12). For (5.14) we take $\frac{\partial^j H}{\partial \gamma^j} = \frac{\partial^j q}{\partial \gamma^j}$, $\frac{\partial^k H}{\partial \varrho^k} = \frac{\partial^k q}{\partial \varrho^k}$; then if we let $c = \pi$, $d = \pi$, $[a = \alpha = \text{arc sin }(1-r),\ b = \beta = \text{arc sin }(1-p)] \to o$, we have

$(5.21) = (1-r)^2 O(1)(1-p)^2 O(1) = o(1)$, $(5.22) = (1-r)^2 O(1)(1-p)^2 O\left(\frac{1}{\beta^{k+3}}\right)o(\beta^{k+1}) = o(1)$,

$$(5.23) = (1-r)^2 O\left(\frac{1}{\alpha^{j+1}}\right)(1-p)^2 O\left(\frac{1}{\beta^{k+3}}\right) o(\alpha^{j+1}\beta^{k+1}) = o(1),$$

$$(5.24) = (1-r)^2 O\left(\frac{1}{\alpha^{j+3}}\right)(1-p)^2 O(1) o(\alpha^{j+1}) = o(1),$$

$$(5.25) = (1-r)^2 O\left(\frac{1}{\alpha^{j+3}}\right)(1-p)^2 o\left(\alpha^{j+1}\int_{\beta}^{\pi}\frac{\varrho^{k+1}}{\varrho^{k+4}}\,d\varrho\right) = o(1),$$

$$(5.26) = (1-r)^2 O(1)(1-p)^2 O\left(\int_{\beta}^{\pi}\frac{\varrho^{k+1}}{\varrho^{k+4}}\,d\varrho\right) = o(1),$$

$$(5.27) = (1-p)^2 O\left(\frac{1}{\beta^{j+3}}\right)(1-r)^2 o\left(\beta^{k+1}\int_{\alpha}^{\pi}\frac{\gamma^{j+1}}{\gamma^{j+4}}\,d\gamma\right) = o(1),$$

$$(5.28) = (1-p)^2 O(1)(1-r)^2 O\left(\int_{\alpha}^{\pi}\frac{\gamma^{j+1}}{\gamma^{j+4}}\,d\gamma\right) = o(1),$$

$$(5.29) = \left(\int_{\beta}^{v}\int_{\alpha}^{\xi} + \int_{\beta}^{v}\int_{\xi}^{\pi} + \int_{v}^{\pi}\int_{\alpha}^{\xi} + \int_{v}^{\pi}\int_{\xi}^{\pi}\right)\ddot{\varphi}_{jk}(x, y\,;\gamma, \varrho)\,\frac{\partial^{j+1}q(r, \gamma)}{\partial\gamma^{j+1}}\,\frac{\partial^{k+1}q(p, \varrho)}{\partial\varrho^{k+1}}\,d\gamma d\varrho$$

$$= (1-r)^2(1-p)^2 o\left(\int_{\beta}^{v}\int_{\alpha}^{\xi}\frac{\gamma^{j+1}\varrho^{k+1}}{\gamma^{j+4}\varrho^{k+4}}\,d\gamma d\varrho\right) + (1-r)^2(1-p)^2 o\left(\int_{\beta}^{v}\int_{\xi}^{\pi}\frac{\varrho^{k+1}}{\varrho^{k+1}}\,d\gamma d\varrho\right) + (1-r)^2(1-p)^2$$

$$o\left(\int_{v}^{\pi}\int_{\alpha}^{\xi}\frac{\gamma^{j+1}}{\gamma^{j+4}}\,d\gamma d\varrho\right) + (1-r)^2(1-p)^2 O(1) = o(1)+o(1)+o(1)+o(1) = o(1),$$ by Lemma 3,

whence $(5.14) = o(1)$.

In consideration of (5.15) we take $\dfrac{\partial^j H}{\partial\gamma^j} = \dfrac{\partial^j K}{\partial\gamma^j}$, $\dfrac{\partial^k H}{\partial\varrho^k} = \dfrac{\partial^k q}{\partial\varrho^k}$; then if we let $c = \pi$,

$$d = \pi,\ [a = \alpha = \text{arc sin}\,(1-r),\ b = \beta = \text{arc sin}\,(1-p)] \to 0,$$

we have $$(5.21) = O(1)(1-p)^2 O(1) O(1) = o(1)$$

$$(5.22) = O(1-p)^2 O\left(\frac{1}{\beta^{k+3}}\right) o(\beta^{k+1}) = o(1),\ (5.25) = O\left(\frac{1}{\alpha^{j+1}}\right)(1-p)^2 O\left(\frac{1}{\beta^{k+3}}\right) o(\alpha^{j+1}\beta^{k+1}) = o(1),$$

$$(5.24) = O\left(\frac{1}{\alpha^{j+1}}\right)(1-p)^2 O(1) o(\alpha^{j+1}) = o(1),\ (5.25) = O\left(\frac{1}{\alpha^{j+1}}\right)(1-p)^2 o\left(\alpha^{j+1}\int_{\beta}^{\pi}\frac{\varrho^{k+1}}{\varrho^{k+4}}\,d\varrho\right) = o(1),$$

$$(5.26) = -\frac{\partial^j K(r, \pi)}{\partial\gamma^j}\left(\int_{\beta}^{v} + \int_{v}^{\pi}\right)\dot{\varphi}_{jk}(x, y\,;\pi, \varrho)\,\frac{\partial^{k+1}q(p, \varrho)}{\partial\varrho^{k+1}}\,d\varrho$$

$$= O(1)(1-p)^2 o\left(\int_{\beta}^{v}\frac{\varrho^{k+1}}{\varrho^{k+4}}\,d\varrho\right) + O(1)(1-p)^2 O(1) = o(1)+o(1) = o(1),$$

$$(5.27) = \frac{\partial^k q(p,\beta)}{\partial\varrho^k}\left(\int_\alpha^\xi + \int_\xi^\pi\right)\dot{\varphi}_{jk}(x, y; \gamma, \beta)\frac{\partial^{j+1}K(r,\gamma)}{\partial\gamma^{j+1}}\,d\gamma$$

$$= (1-p)^2 O\left(\frac{1}{\beta^{k+3}}\right) o\left(\beta^{k+1}\int_\alpha^\xi \gamma^{j+1}\begin{bmatrix}\dfrac{\sin\gamma}{(1-r)^{j+3}} \\ \dfrac{1}{(1-r)^{j+2}}\end{bmatrix} d\gamma\right) + (1-p)^2 o O\left(\frac{1}{\beta^{k+2}}\right) o\left(\beta^{k+1}\int_\xi^\pi d\gamma\right)$$

$$= o(1) + o(1) = o(1),$$

$$(5.28) = -\frac{\partial^k q(p,\pi)}{\partial\varrho^k}\left(\int_\alpha^\xi + \int_\xi^\pi\right)\dot{\varphi}_{jk}(x, y; \gamma\,\pi)\frac{\partial^{j+1}K(r,\gamma)}{\partial\gamma^{j+1}}\,d\gamma$$

$$= (1-p)^2 O(1) o\left(\int_\alpha^\xi \gamma^{j+1}\begin{bmatrix}\dfrac{\sin\gamma}{(1-r)^{j+3}} \\ \dfrac{1}{(1-r)^{j+2}}\end{bmatrix} d\gamma\right) + (1-p)^2 O(1) O\left(\int_\xi^\pi d\gamma\right) = o(1) + o(1) = o(1),$$

$$(5.29) = \left(\int_\beta^v\int_\alpha^\xi + \int_\beta^v\int_\xi^\pi + \int_v^\pi\int_\alpha^\xi + \int_v^\pi\int_\xi^\pi\right)\ddot{\varphi}_{jk}(x, y; \gamma, \varrho)\frac{\partial^{j+1}K(r,\gamma)}{\partial\gamma^{j+1}}\,\frac{\partial^{k+1}q(p,\varrho)}{\partial\varrho^{k+1}}\,d\gamma d\varrho$$

$$= (1-\varrho)^2 o\left(\int_\beta^v\int_\alpha^\xi \gamma^{j+1}\varrho^{k+1}\begin{bmatrix}\dfrac{\sin\gamma}{(1-r)^{j+3}} \\ \dfrac{1}{(1-r)^{j+2}}\end{bmatrix}\frac{1}{\varrho^{k+1}}\,d\gamma d\varrho\right) + (1-p)^2 o\left(\int_\beta^v\int_\xi^\pi \frac{\varrho^{k+1}}{\varrho^{k+4}}\,d\gamma d\varrho\right) +$$

$$(1-p)^2 o\left(\int_\gamma^\pi\int_\alpha^\xi \gamma^{j+1}\begin{bmatrix}\dfrac{\sin\gamma}{(1-r)^{j+3}} \\ \dfrac{1}{(1-r)^{j+2}}\end{bmatrix} d\gamma d\varrho\right) + (1-p)^2 O(1) = o(1) + o(1) + o(1) + o(1) = o(1),$$

by Lemmas 1 and 8, whence (5.15) $= o(1)$. (5.16) $= o(1)$ by analogy with (5.15). This proves the Lemma.

Proof of the Theorem. Upon taking partial derivatives irrespective of order, we have

$$\frac{\partial^{j+k}A(x, y; r, p)}{\partial x^j dy^k} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\frac{\partial^j K(r, u-x)}{\partial x^j}\,\frac{\partial^k K(p, v-y)}{\partial y^k}\,dudv$$

$$= \frac{(-1)^{j+k}}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\frac{\partial^j K(r, u-x)}{\partial u^j}\,\frac{\partial^k(p, v-y)}{\partial v^k}\,dudv = \frac{(-1)^{j+k}}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x+\gamma, y+\varrho)\frac{\partial^j K(r,\gamma)}{\partial\gamma^j}$$

$$\times\frac{\partial^j K(p,\varrho)}{\partial\varrho^k}\,d\gamma d\varrho = \frac{(-1)^{j+k}}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi} [f(x+\gamma, y+\varrho) - f(x+\gamma, y) - f(x, y+\varrho) + f(x,y)]\frac{\partial^j K(r,\gamma)}{\partial\gamma^j}$$

$$\times\frac{\partial^k K(p,\varrho)}{\partial\varrho^k}\,d\gamma d\varrho = \frac{(-1)^{j+k}}{\pi^2}\int_0^\pi\int_0^\pi \varphi_{jk}(x, y; \gamma, \varrho)\frac{\partial^j K(r,\gamma)}{\partial\gamma^j}\,\frac{\partial^k K(p,\varrho)}{\partial\varrho^k}\,d\gamma d\varrho$$ because of the periodicity of f and K and by Lemma 2,

We have, from Lemma 4

$$\frac{\partial^{j+k}A(x,\, y\,;\, r\ p)}{\partial x^j \partial y^k} - \frac{(-1)^{j+k}}{4\pi^2} \int_\beta^\pi \int_\alpha^\pi \varphi_{jk}(x,\, y\,;\, \gamma,\, \varrho)\, \frac{d^j \cot \frac{1}{2}\gamma}{d\gamma^j}\, \frac{d^k \cot \frac{1}{2}\varrho}{d\varrho^k}\, d\gamma d\varrho$$

$$= \frac{(-1)^{j+k}}{\pi^2} \left(\int_0^\beta \int_0^\alpha + \int_0^\beta \int_\alpha^\pi + \int_\beta^\pi \int_0^\alpha \right) \varphi_{jk}(x,\, y\,;\, \gamma,\, \varrho) \frac{\partial^j K(r,\, \gamma)}{\partial \gamma^j}\, \frac{\partial^k K(p,\, \varrho)}{\partial \varrho^k}\, d\gamma d\varrho$$

$$- \frac{(-1)^{j+k}}{\pi^2} \int_\beta^\pi \int_\alpha^\pi \varphi_{jk}(x),\, y\,;\, \gamma,\, \varrho) \left[\frac{\partial^j q(r,\, \gamma)}{\partial \gamma^j}\, \frac{\partial^k q(p,\, \varrho)}{\partial \varrho^k} + \frac{\partial^j K(r,\, \gamma)}{\partial \gamma^j}\, \frac{\partial^k q(p,\, \varrho)}{\partial \varrho^k} \right.$$

$$\left. + \frac{\partial^j q(r,\, \gamma)}{\partial \gamma^j}\, \frac{\partial^k K(p,\, \varrho)}{\partial \varrho^k} \right] d\gamma d\varrho,$$

the right member of which tends to 0 as $[\alpha = \text{arc sin } (1-r),\ \beta = \text{arc sin } (1-p)]$ tends to 0 by Lemma 5, thereby proving the theorem.

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ON THE STRESSES DUE TO A NUCLEUS IN THE FORM OF A CENTRE OF ROTATION IN AN INFINITE ELASTIC SOLID WITH SPHERICAL INCLUSION

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1. **Introduction.** In this paper stresses in an infinite elastic solid due to a nucleus in the form of a centre of rotation at a finite distance from a spherical inclusion have been studied. To begin with a general solution has been obtained with an elastic inclusion and hence particular cases of rigid inclusion and cavity have been discussed. Numerical values of the stresses have also been given for a spherical elastic inclusion when $\mu_2/\mu_1 = 2$ and $c = 5$.

2. **Formulation of the problem and boundary conditions.** The axis of rotation is taken to be the z-axis with origin at the centre of the spherical inclusion. The centre of rotation is at a distance c from the origin where $c > a$, a being the radius of the sphere.

In such cases the resultant displacement due to the centre of rotation is given by (Love, 1944).

$$\frac{P}{4\pi\mu_1}\left\{\frac{\partial}{\partial y}\left(\frac{1}{\varrho}\right),\ -\frac{\partial}{\partial x}\left(\frac{1}{\varrho}\right),\ 0\right\}$$

or

$$\frac{P}{4\pi\mu_1}\left\{-\frac{y}{\varrho^3},\ \frac{x}{\varrho^3},\ 0\right\} \tag{1}$$

where $2P$ represents the moment of a couple about z-axis and μ_1 the modulus of rigidity of the infinite solid. Using spherical polar coordinates, the only non-vanishing component of displacement is given by

$$u_{\varphi_1} = -\frac{P}{4\pi\mu_1}\ \frac{r\sin\theta}{\varrho^3}$$

$$= \frac{P}{4\pi\mu_1}\cdot\frac{1}{c}\cdot\frac{\partial}{\partial\theta}\left\{r^2 - 2rc\cos\theta + c^2\right\}^{-\frac{1}{2}} \tag{2}$$

Assuming u_r and u_θ to be zero and $u_\varphi = w$, independent of φ, two equations of equilibrium are identically satisfied and the third one takes the form

$$r^2\frac{\partial^2\mu_\varphi}{\partial r^2} + 2r\,\frac{\partial\mu_\varphi}{\partial r} + \frac{\partial}{\partial\theta}\left(\frac{\partial u_\varphi}{\partial\theta} + u_\varphi\cot\theta\right) = 0 \tag{3}$$

The general solution of the equation is given by

$$w = u_\varphi = \sum_{n=1}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)\frac{d}{d\theta}[P_n(\cos\theta)] \tag{4}$$

when A_n and B_n are constants and $P_n(\cos\theta)$ is the Legendre's function of the first kind and nth. degree. For the present problem we assume the solution in the form

$$w_1 = -\frac{P}{4\pi\mu_1}\cdot\frac{r\sin\theta}{(r^2-2rc\cos\theta+c^2)^{3/2}} + \sum_{n=1}^{\infty}\frac{B}{r^{n+1}}\frac{d}{d\theta}[P_n(\cos\theta)], \tag{5}$$

outside the inclusion

and

$$w_2 = \sum_{n=1}^{\infty} A_n r^n \frac{d}{d\theta}[P_n(\cos\theta)], \tag{6}$$

inside the inclusion.

The boundary conditions to be satisfied in this case are

$$w_1 = w_2, \tag{7}$$

and

$$\widehat{r\varphi}_1 = \widehat{r\varphi}_2, \text{ on the surface of the inclusion.} \tag{8}$$

Suffixes 1 and 2 denote expressions for outside and inside the inclusion.

3. Solution of the problem. Stresses outside the inclusion are

$$\widehat{r\varphi}_1 = \mu_1\left[\frac{\partial w_1}{\partial r} - \frac{w_1}{r}\right] = \frac{3P}{4\pi}\cdot\frac{r\sin\theta(r-c.\cos\theta)}{(r^2-2cr\cos\theta+c^2)^{5/2}} - \mu_1\sum_{n=1}^{\infty}\frac{(n+2)Bn}{r^{n+2}}\frac{d}{d\theta}[P_n(\cos\theta)] \tag{9}$$

and

$$\widehat{\theta\varphi}_1 = \frac{\mu_1}{r}\left[\frac{\partial w_1}{\partial\theta} - w_1\cot\theta\right]$$

$$= \frac{3P}{4\pi}\frac{cr\sin^2\theta}{(r^2-2rc\cos\theta+c^2)^{5/2}} + \mu_1\sum_{n=1}^{\infty}\frac{B_n}{r^{n+2}}\left\{\frac{d^2}{d\theta^2}[P_n(\cos\theta)] - \cot\theta\frac{d}{d\theta}[P_n(\cos\theta)]\right\} \tag{10}$$

Stresses inside the inclusion are

$$\widehat{r\varphi}_2 = \mu_2\sum_{n=1}^{\infty}(n-1)A_n r^{n-1}\frac{d}{d\theta}[P_n(\cos\theta)] \tag{11}$$

and

$$\widehat{\theta\varphi}_2 = \mu_2\sum_{n=1}^{\infty}A_n r^{n-1}\left\{\frac{d^2}{d\theta^2}[P_n(\cos\theta)] - \cot\theta\frac{d}{d\theta}[P_n(\cos\theta)]\right] \tag{12}$$

If we take $a = 1$, on the surface of the inclusion, we have

$$w_1 = -\frac{P}{4\pi\mu_1}\frac{\sin\theta}{(1+c^2-2c\cos\theta)^{3/2}} + \sum_{n=1}^{\infty}B_n\frac{d}{d\theta}[P_n(\cos\theta)],$$

$$= \frac{P}{4\pi\mu_1}\sum_{n=1}^{\infty}\frac{1}{c^{n+2}}\,\frac{d}{d\theta}[P_n(\cos\theta)] + \sum_{n=1}^{\infty}B_n\frac{d}{d\theta}[P_n(\cos\theta)], \tag{13}$$

$$\widehat{r\varphi}_1 = \frac{3P}{4\pi}\,\frac{\sin\theta\,(1-c\cos\theta)}{(1-2c\cos\theta+c^2)^{5/2}} - \mu_1\sum_{n=1}^{\infty}(n+2)B_n\frac{d}{d\theta}[P_n(\cos\theta)],$$

$$= \frac{P}{4\pi}\sum_{n=1}^{\infty}\frac{n-1}{c^{n+2}}\,\frac{d}{d\theta}[P_n(\cos\theta)] - \mu_1\sum_{n=1}^{\infty}(n+2)B_n\frac{d}{d\theta}[P_n(\cos\theta)]. \tag{14}$$

and
$$w_2 = \sum_{n-1}^{\infty}A_n\frac{d}{d\theta}[P_n(\cos\theta)]. \tag{15}$$

$$\widehat{r\varphi}_2 = \mu_2\sum_{n=1}^{\infty}(n-1)A_n\frac{d}{d\theta}[P_n(\cos\theta)] \tag{16}$$

Boundary conditions (7) and (8) will be satisfied if

$$\frac{P}{4\pi\mu_1}\,\frac{1}{c^{n+2}} + B_n = A_n \tag{17}$$

$$\frac{P}{4\pi}\cdot\frac{(n-1)}{c^{n+2}} - \mu_1(n+2)B_n = \mu_2(n-1)A_n \tag{18}$$

whence
$$A_n = \frac{P}{4\pi}\cdot\frac{2n+1}{c^{n+2}\{\mu_1(n+\quad+\mu_2(n-1)\}} \tag{19}$$

$$B_n = \frac{P}{4\pi}\cdot\frac{n-1}{c^{n+2}}\cdot\frac{\mu_1-\mu_2}{\mu_1}\cdot\frac{1}{\mu_1(n+2)+\mu_2(n-1)} \tag{20}$$

Displacement and stress components on the common boundary are obtained from (15) and (16) or from (13) and (14) on substituting the values of constants from (19) and 20) in the form

$$w = \frac{P}{4\pi\mu_2}\sum_{n=1}^{\infty}\frac{2n+1}{c^{n+2}\{(\mu_1/\mu_2)(n+2)+(n-1)\}}\,\frac{d}{d\theta}[P_n(\cos\theta)] \tag{21}$$

$$\widehat{r\varphi} = \frac{P}{4\pi}\sum_{n=1}^{\infty}\frac{(n-1)(2n+1)}{c^{n+2}\{(\mu_1/\mu_2)(n+2)+(n-1)\}}\,\frac{d}{d\theta}[P_n(\cos\theta)] \tag{22}$$

4. Discussion. (1) When the inclusion is rigid, $\mu_2\to\infty$ and $\mu_1 = \mu$ (say)

$$\left.\begin{aligned} A_n &= 0 \\ B_n &= -\frac{P}{4\pi\mu}\cdot\frac{1}{c^{n+2}} \end{aligned}\right\} \tag{23}$$

Displacement and stress components on the surface of the inclusion are then given by

$$w = 0, \tag{24}$$

$$\widehat{r\varphi} = \frac{P}{4\pi}\sum_{n=1}^{\infty}\frac{2n+1}{c^{n+2}}\;\frac{d}{d\theta}[P_n(\cos\theta)], \tag{25}$$

and $$\widehat{\varphi\theta} = \frac{P}{4\pi}\left[\frac{3c\sin^2\theta}{(1-2c\cos\theta+c^2)^{5/2}}+\sum_{n+1}^{\infty}\frac{1}{c^{n+2}}\left\{2\cot\theta\frac{d}{d\theta}[P_n(\cos\theta)]+n(n+1)\sin\theta P_n(\cos\theta)\right\}\right. \tag{26}$$

(2) In the case of a cavity, $\mu_2 \to 0$ and $\mu_1 = \mu$ (say),

$$A_n = \frac{P}{4\pi}\,\frac{2n+1}{c^{n+2}\mu(n+2)} \tag{27}$$

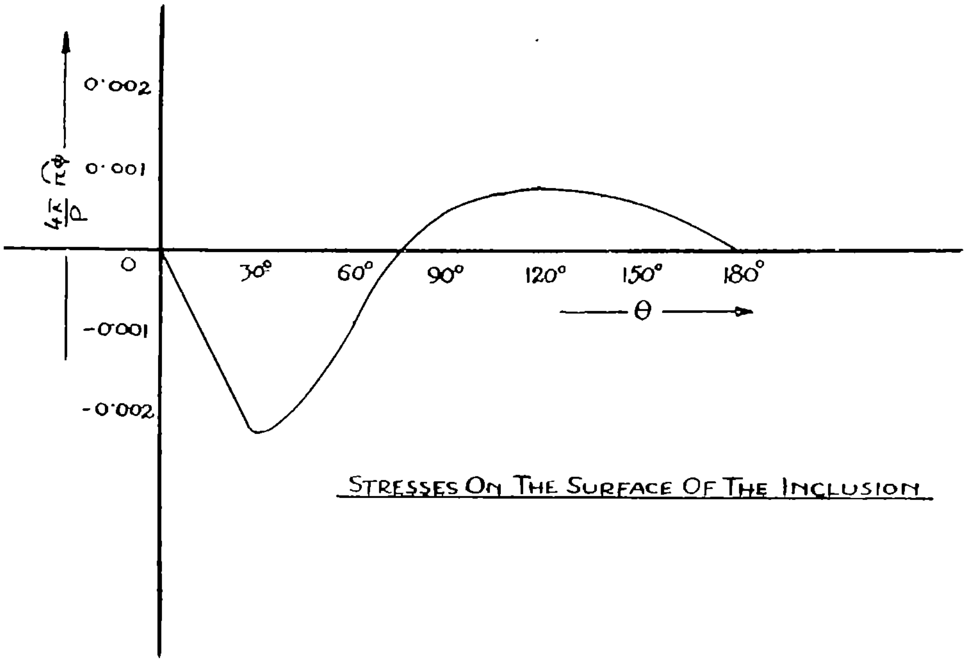
$$B_n = \frac{P}{4\pi}.\frac{n-1}{c^{n+2}}.\frac{1}{\mu(n+2)} \tag{28}$$

The displacement and stress components on the surface of the cavity are given by

$$w_1 = w = \frac{P}{4\pi\mu}\sum_{n=1}^{\infty}\frac{1}{c^{n+2}}.\frac{2n+1}{n+2}\,\frac{d}{d\theta}[P_n(\cos\theta)] \tag{29}$$

and $$\widehat{\theta\varphi_1} = \widehat{\theta\varphi} = \frac{P}{4\pi}\left[\frac{3c\sin^2\theta}{(1-2c\cos\theta+c^2)^{5/2}}-\sum_{n=1}^{\infty}\frac{n-1}{(n+2)c^{n+2}}\left\{2\cot\theta\frac{d}{d\theta}[P_n(\cos\theta)]\right.\right.$$

$$\left.+n(n+1)\sin\theta \mathrm{P}_n(\cos\theta)\right\} \tag{30}$$



STRESSES ON THE SURFACE OF THE INCLUSION

5. Numerical calculations. Numerical results have been obtained in the case of a spherical elastic inclusion when $\mu_2/\mu_1 = 2$ and $c = 5$. Stress component $\widehat{r\varphi}$ on the surface of the inclusion is shown in the figure above. There are three points on the inclusion at 0°, 75° (nearly), 180° where the stress is zero. The maximum stress is obtained at near about 30°.

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CERTAIN TRANSFORMATIONS ON UNILATERAL AND BILATERAL OPERATIONAL CALCULUS

By

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1. In this paper the author has obtained certain unilateral and bilateral relations and has contructed some examples in the case of one variable which are believed to be new. The author has also obtained Chatterjee's theorem (Chatterjee, 1958) on exponential transformation as a particular case.

2. Let $f(x, y)$ be a function of two variables x and y, defined for all pairs in $0 < x < \infty$ and $0 < y < \infty$ and integrable in every finite rectangle.

$$R_{XY}:\ 0 < x < X,\ 0 < y < Y.$$

If for a pair of complex quantities p and q, the limit of the double integral

$$\lim_{\substack{X \to \infty \\ Y \to \infty}} \int_0^X \int_0^Y e^{-px-qy} U(x, y) f(x, y)\,dx\,dy \tag{1}$$

where $U(x, y)$ is, Heaviside's unit function, exists, then evidently the double integral

$$L^2_{II}\{f\} = \int_0^\infty \int_0^\infty e^{-px-qy} f(x, y)\,dx\,dy \tag{2}$$

exists and we shall call it the two dimensional Laplace integral for the pair of values p and q. If $L^2_{II}\{f\}$ exists, not only for one pair of values p and q but for every pair of values (p, q) in certain associated regions of the complex p, q planes, then we call it two dimensional Laplace transform of $f(x, y)$.

If (2) converges absolutely for a complex pair of values p and q then it converges absolutely in the complex half planes

$$\mathbf{R}(p) \geq \mathbf{R}(p_0),\ \mathbf{R}(q) \geq \mathbf{R}(q_0).$$

We restrict ourselves to absolute convergence. The determination of the complex domain of absolute convergence depends only on the real values of p and q. If it converges absolutely for at least one pair of values p and q, then in general there is no unique pair of convergence half planes but a whole family of "associated convergence half planes". We can illustrate it geometrically in the following way.*

For every fixed $p = p_0 > A_1$, there is a convergence abcissa $a_2(p_0)$ of q. $L^2_{II}\{f\}$ converges for $p = p_0$, $q > a_2(p_0)$ and diverges for $p = p_0$ and $q < a_2(p_0)$. The vertical $p = p_0$ is divided into a convergent and divergent ray by the point whose ordinate is

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* In the next paper.

$a_2(p_0)$. If $p = p_1 > p_0$, then $L^2_{II}\{f\}$ converges for $p = p_1$, $q > a_2(p_0)$ and the true convergence-abscissa $a_2(p_1) \leq a_2(p_0)$ for $p_1 > p_0$. Thus $a_2(p)$ is a monotonic decreasing function and as p increases, it may tend to a finite value or may be equal to a finite constant say A_2 from a certain point onward or may tend to $-\infty$. The inverse of $q = a_2(p)$ is $p = a_1(q)$ for $q > A_2$ where A_2 is a number similar to A_1 with respect to q and under certain circumstances A_1 and A_2 may tend to $-\infty$. For the q values A_1 plays the same part as A_2 for p-values.

If we draw the curves representing $q = a_2(p)$, then they will have different shapes according as $a_2(p)$ tends to a finite constant or is equal to a finite constant or tends to $-\infty$. Convergence occurs at all points inside the curves and divergence at all points outside. For points on the curve, convergence or divergence may occur. The accurate domain of absolute convergence for real (p, q) is the interior of a curve called the convergence charcteristic-curve. In this way, the correspondence of convergence half-planes is determined.

In bilateral transform, we have four integrals

$$I_1 = \int_0^\infty \int_0^\infty e^{-px-qy} f(x, y)\,dx dy \qquad \text{converging for } p \geq p_0,\ q \geq q_0,$$

$$I_2 = \int_0^\infty \int_{-\infty}^0 e^{-px-qy} f(x, y)\,dx dy \qquad \text{converging for } p \geq p_0,\ q \leq q_0,$$

$$I_3 = \int_{-\infty}^0 \int_0^\infty e^{-px-qy} f(x, y)\,dx dy \qquad \text{converging for } p \leq p_0,\ q \geq q_0,$$

$$I_4 = \int_{-\infty}^0 \int_{-\infty}^0 e^{-px-qy} f(x, y)\,dx dy \qquad \text{converging for } p \leq p_0,\ q \leq q_0.$$

All the integrals I_1, I_2, I_3 I_4, have one-sided double Laplace integrals of paramaters (p, q), $(p, -q)$ $(-p, q)$ and $(-p, -q)$ respectively.

If the function $f(x, y)$ be such that the four convergence domains when represented separately by the convergence characteristic curves over-lap and have a common part Ω, then Ω will represent the accurate domain of the absolute convergence of the double integral

$$\int_{-\infty}^\infty \int_{-\infty}^\infty e^{-px-qy} f(x, y)\,dx dy.$$

If there is no common domain, then $f(x, y)$ has no two-sided double Laplace transform. For points on the boundary of Ω, there may be either convergence or divergence.

To intrepret, we consider the greatest rectangle contained in Ω whose adjacent sides are parallel to the p and q axes respectively. Then the p values corresponding to points on the side of the rectangle parallel to the p-axis combined with the q values corresponding to points on the adjacent side of the same rectangle parallel to the q-axis will give the absolute convergence of $L^2_{II}\{f\}$. These p and q values in the complex planes correspond to a pair of "associated convergence strips". There is no unique pair of "associated convergence strips" but a whole family of pairs of associated convergence strips. If the convergence-characteristic curves of I_1, I_2, I_3, I_4, consist of horizontal and vertical lines only, the common portion Ω has the shape of a rectangle and we have a single pair of convergence strips.

3. **Theorem 1.** *(i) Let* $F(p_1, p_2) \doteqdot (x_1, x_2)^{1/k} f(x_1, x_2) U(x_1, x_2)$ *in a pair of associated halfplanes* H_{p_1}, H_{p_2} *say.*

(ii) $e^{-\lambda_i \psi_i(p_i)}\{\psi_i(p_i)\}^m \doteqdot x_i h_i(\lambda_i, x_i), \ \lambda_i > 0, \qquad i = 1, 2$

in the half planes $D_{p_i} (i = 1, 2)$ *say, where* D_{p_i} *represents the plane,* $\mathrm{R}(p_i) > 0$ *and* $e^{-\lambda_i \psi_i(p_i)}\{\psi_i(p_i)\}^m$ *is bounded and integrable in* $(0, \infty)$ *in* p_i;

(iii) $f(x_1, x_2)(x_1 x_2)^{1/k-1}$ *is absolutely integrable in* $(0, \infty)$ *in* x_1 *and* x_2.

Then

$$K(p_1, p_2) = \frac{p_1 p_2}{k^2} \int_0^\infty \int_0^\infty h_1(p_1, x_1) h_2(p_2, x_2) F(x_1, x_2) dx_1 dx_2$$

$$\doteqdot (x_1 x_2)^m f\{\varphi_1^k(x_1), \varphi_2^k(x_2)\} \varphi_1'(x_1) \varphi_2'(x_2)$$

provided that the integral on the right is absolutely convergent in a pair of associated convergence strips,

(iv) $\varphi_i(x_i)$ *is differentiable in* x_i,

(v) $\varphi_i(x_i)$ *tends to zero or infinity as* x_i *tends to zero and tends to infinity or zero as* x_i *tends to infinity,*

(vi) $x_i = \varphi^{-1}(s_i^{1/k}) = \psi_i(s_i)$, φ_i^{-1} *being the inverse of* φ_i. *The latter condition and (iv) can be replaced by the condition that* $\varphi_i(x_i)$ *is monotonic.*

Proof. Consider the integral

$$I = p_1 p_2 \int_0^\infty \int_0^\infty e^{-p_1 x_1 - p_2 x_2} (x_1 x_2)^m f\{\varphi_1^k(x_1), \varphi_2^k(x_2)\} \varphi_1'(x_1) \varphi_2'(x_2) dx_1 dx_2. \tag{1}$$

Let us write $s_i = \varphi_i^k(x_i)$.

Then $k \varphi_i^{k-1}(x_i) \varphi_i'(x_i)\, dx_i = ds_i$,

$$I = \frac{p_1 p_2}{k^2} \int_0^\infty \int_0^\infty e^{-p_1 \psi_1(s_1) - p_2 \psi_2(s_2)} \frac{\{\psi_1(s_1), \psi_2(s_2)\}^m}{s_1 s_2} f(s_1, s_2)(s_1 s_2)^{1/k} ds_1 ds_2 \tag{2}$$

Now making use of (ii) we get

$$I = \frac{p_1 p_2}{k^2} \int_0^\infty \int_0^\infty (s_1 s_2)^{1/k} \frac{f(s_1, s_2)}{s_1 s_2} ds_1 ds_2 \left\{ s_1 s_2 \int_0^\infty \int_0^\infty e^{-s_1 x_1 - s_2 x_2} x_1 x_2 h_1(p_1, x_1) h_2(p_2, x_2) dx_1 dx_2 \right\}$$

On changing the orders of integrations which is permissible in view of absolute convergence of the x- and s-integrals, we have

$$I = \frac{p_1 p_2}{k^2} \int_0^\infty \int_0^\infty x_1 x_2 h_1(p_1, x_1) h_2(p_2, x_2) dx_1 dx_2 \left\{ \int_0^\infty \int_0^\infty (s_1 s_2)^{1/k} e^{-s_1 x_1 - s_2 x_2} f(s_1, s_2) ds_1 ds_2 \right\}$$

$$= \frac{p_1 p_2}{k^2} \int_0^\infty \int_0^\infty h_1(p_1, x_1) h_2(p_2, x_2) F(x_1, x_2) dx_1 dx_2$$

$$= K(p_1, p_2) \doteqdot (x_1 x_2)^m f\{\varphi_1{}^k(x_1), \varphi_2{}^k(x_2)\} \varphi_1'(x_1) \varphi_2'(x_2) \tag{3}$$

We have established the theorem under the most stringent conditions. Many of the conditions can now be waived. It is sufficient if (*i*) $h_1(p_1, x_1)\, h_2(p_2, x_2) F(x_1, x_2)$ is absolutely integrable in $(0, \infty)$ in each of the variables x_1, and x_2 in a pair of associated convergence strips,

and $$(x_1 x_2)^m f\{\varphi_1{}^k(x_1), \varphi_2{}^k(x_2)\} \varphi_1'(x_1) \varphi_2'(x_2)$$

being bounded and integrable in $(0, \infty)$ in each of the variables x_1, and x_2.

Cor. 1. In particular let $\varphi_i(x_i) = x_i$, $(i = 1, 2)$. Then $\psi_i(s_i) = s_i^{1/k}$

Also let $k = 2$, $m = \nu$.

Then $$F(p_1, p_2) \doteqdot (x_1 x_2)^{\frac{1}{2}} f(x_1, x_2) U(x_1, x_2)$$

since $e^{-\lambda_i (p_i) 1/k} (p_i)^{m/k} \doteqdot x_i h_i(\lambda_i, x_i)$, assuming that $1/p_i \doteqdot t_i$

we get $$e^{-\lambda_i (p_i)^{\frac{1}{2}}} (p_i)^{\nu/2} \doteqdot t_i h_i(\lambda_i, t_i)$$

But $$e^{-\lambda_i (p_i)^{\frac{1}{2}}} (p_i{}^{\nu/2}) \doteqdot \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (2t_i)^{-\nu/2} e^{-\lambda_i{}^2/(8t_i)} D_{\nu-1}\left(\frac{\lambda_i}{(2t_i)^{\frac{1}{2}}}\right)$$

Therefore by the Uniqueness theorem,

$$t_i h_i(\lambda_i, t_i) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (2t_i)^{-\nu/2} e^{-\lambda_i{}^2/(8t_i)} D_{\nu-1}\left(\frac{\lambda_i}{(2t_i)^{\frac{1}{2}}}\right)$$

Therefore $$I = \frac{p_1 p_2}{2\pi} \int_0^\infty \int_0^\infty \frac{(4t_1 t_2)^{-\nu/2}}{t_1 t_2} e^{-p_1{}^2/(8t_1) - p_2{}^2/(8t_2)} D_{\nu-1}\left(\frac{p_1}{(2t_1)^{\frac{1}{2}}}\right) D_{\nu-1}\left(\frac{p_2}{(2t_2)^{\frac{1}{2}}}\right) \times F(t_1, t_2) dt_1 dt_2$$

$$\doteqdot (x_1 x_2)^\nu f(x_1{}^2, x_2{}^2)$$

Cor. 2. Let $k = -1$, $m = \nu$.

since $$\frac{e^{-\lambda_i/p_i}}{p_i{}^\nu} \doteqdot (\lambda_i)^{-\nu/2} (t_i)^{\nu/2} J_\nu(2(\lambda_i t_i)^{\frac{1}{2}})$$

we have by Uniquenss theorem $t_i h_i(\lambda_i, t_i) = \lambda_i{}^{-\nu/2} (t_i)^{\nu/2} J_\nu(2(\lambda_i t_i)^{\frac{1}{2}})$.

$$I = p_1p_2\int_0^\infty\int_0^\infty (p_1p_2)^{-\nu/2}(t_1t_2)^{\nu/2-1}J_\nu(2(p_1t_1)^{\frac{1}{2}})J_\nu(2(p_2t_2)^{\frac{1}{2}})F(t_1,\, t_2)\,dt_1dt_2$$

$$\doteqdot (x_1x_2)^\nu f\left(\frac{1}{x_1},\, \frac{1}{x_2}\right).$$

4. As applications of theorem I, for one variable only, we shall find some one-dimensional correspondences.

(*a*) Let us take the operational relation

$$p^{-\lambda}e^{-a/p^\mu} \doteqdot x^\lambda J_\lambda{}^\mu(ax^\mu)U(x),\quad a>0,\ \mathbf{R}(p)>0,\ \mu>0,\ \mathbf{R}(\lambda)>-1$$

we have, when $\mu = 1,\quad k = 2$

$$F(p) = p^{-\lambda}e^{-a/p} \doteqdot \left(\frac{x}{a}\right)^{\lambda/2} J_\lambda(2(ax)^{\frac{1}{2}})$$

since $F(p) \doteqdot (x)^{\frac{1}{2}}f(x)$

$\therefore$ $f(x) \equiv \dfrac{x^{(\lambda-1)/2}}{a^{\lambda/2}}J_\lambda(2(ax)^{\frac{1}{2}}).$

by the Uniqueness theorem. On substituting for $F(t)$ and $f(x)$, we get on simplication,

when $\nu = 0$ $$\frac{p}{(2\pi)^{\frac{1}{2}}}\int_0^\infty e^{-p^2t/8}D_{-1}\left(p\left(\frac{t}{2}\right)^{\frac{1}{2}}\right)t^{\lambda-1}e^{-at}dt \doteqdot \frac{x^{\lambda-1}}{a^{\lambda/2}}J_\lambda(2(ax^{\frac{1}{2}}))$$

(*b*) Let $F(p) = p^{-\lambda}e^{-a/p} \doteqdot \left(\dfrac{x}{a}\right)^{\lambda/2}J_\lambda(2(ax)^{\frac{1}{2}}).$

when $\nu = 0,\ k = -1$

then $$\frac{1}{x}f(x) = \left(\frac{x}{a}\right)^{\lambda/2}J_\lambda(2(ax)^{\frac{1}{2}}).$$

or $$f(x) = \frac{x^{\lambda/2+1}}{a^{\lambda/2}}J_\lambda(2(ax)^{\frac{1}{2}})$$

Therefore $-pa^{\lambda/2}\displaystyle\int_0^\infty J_0\left(2\left(\frac{a}{t}\right)^{\frac{1}{2}}\right)t^{\lambda-1}e^{-pt}dt \doteqdot x^{-\lambda/2-1}J_\lambda\left(2\left(\frac{a}{x}\right)^{\frac{1}{2}}\right),\ \mathbf{R}(\lambda)>-\frac{1}{2}$

(*c*) Let $F(p) = \dfrac{e^{-(ap)^{\frac{1}{2}}}}{(p)^{\frac{1}{2}}} \doteqdot \left\{2\left(\dfrac{x}{\pi}\right)^{\frac{1}{2}}e^{-a/(4x)} - a^{\frac{1}{2}}\operatorname{erfc}\left(\dfrac{a^{\frac{1}{2}}}{2x^{\frac{1}{2}}}\right)\right\}U(x)$

$$R(p)>0,\quad a>0$$

Then $$x^{\frac{1}{2}}f(x) = \left\{2\left(\frac{x}{\pi}\right)^{\frac{1}{2}}e^{-a/4x} - a^{\frac{1}{2}}\operatorname{erfc}\left(\frac{a^{\frac{1}{2}}}{2x^{\frac{1}{2}}}\right)\right\}$$

Therefore $$f(x^2) = \frac{1}{x}\left\{2\frac{x}{\pi^{\frac{1}{2}}}e^{-a/4x^2} - a^{\frac{1}{2}}\operatorname{erfc}\left(\frac{a^{\frac{1}{2}}}{2x}\right)\right\}$$

when $k = 2,\ \nu = 0$

hence $$\frac{p}{(2\pi)^{\frac{1}{2}}}\int_0^\infty \frac{e^{-p^2/8t}}{t^{3/2}}\,D_{-1}\left(\frac{p}{(2t)^{\frac{1}{2}}}\right)e^{-(at)^{\frac{1}{2}}}dt \doteqdot \left\{\frac{2}{(\pi)^{\frac{1}{2}}}\,e^{-a/(4x^2)} - \frac{a^{\frac{1}{2}}}{x}\operatorname{erfc}\left(\frac{a^{\frac{1}{2}}}{2x}\right)\right\}$$

(*d*) Let $$F(p) = (ap^{\frac{1}{2}})^{2m+n+2}K_n(ap^{\frac{1}{2}}) \doteqdot \frac{(-1)^{m+n}m\,!}{(m+n)\,!}2^{2m+n+1}e^{-a^2/4x}L^{(n)}{}_{m+n}\left(\frac{a^2}{4x}\right)U(x)$$

when $\nu = 0,\ k = 2$

then $$f(x) = \frac{(-1)^{m+n}m\,!}{(m+n)\,!}2^{2m+n+1}x^{-\frac{1}{2}}e^{-a^2/(4x)}L^{(n)}{}_{m+n}\left(\frac{a^2}{4x}\right)U(x)\,;$$

after some reductions, we get

$$\frac{p}{(2\pi)^{\frac{1}{2}}}\int_0^\infty e^{-p^2/(8t)}D_{-1}\left(\frac{p}{(2t)^{\frac{1}{2}}}\right)t^{m+n/2}K_n(at^{\frac{1}{2}})dt \doteqdot$$
$$\left(\frac{2}{a}\right)^{2m+n+1}\frac{1}{ax}\frac{(-1)^{m+n}m\,!}{(m+n)\,!}e^{-a^2/(4x^2)}L^{(n)}{}_{m+n}\left(\frac{a^2}{4x^2}\right)U(x)$$

(*e*) Similarly, when $\nu = 0,\ k = -1$

$$-p\int_0^\infty J_0(2(pt)^{\frac{1}{2}})t^{m+n/2}K_n(at^{\frac{1}{2}})dt \doteqdot \frac{1}{ax}\left(\frac{2}{a}\right)^{2m+n+1}\frac{(-1)^{m+n}m\,!}{(m+n)\,!}e^{-a^2x/4}L^{(n)}{}_{m+n}\left(\frac{a^2x}{4}\right),\ m > -1$$

(*f*) Let $$F(p) = e^{-(a+b)/p}I_\nu\left(\frac{2(ab)^{\frac{1}{2}}}{p}\right) \doteqdot J_\nu(2(ax)^{\frac{1}{2}})J_\nu(2(bx)^{\frac{1}{2}})U(x).$$

$\mathbf{R}(p),\ \mathbf{R}(\nu) > 0$ and $a \neq b$ a and b are positive.

Therefore, $$f(x^2) = (1/x)J_\nu(2ax^{\frac{1}{2}})J_\nu(2b^{\frac{1}{2}}x)U(x).$$

Hence from the theorem, we get, when $k = 2$

$$\frac{p}{(2\pi)^{\frac{1}{2}}}\int_0^\infty e^{-p^2/(8t)}t^{-1}D_{-1}\left(\frac{p}{(2t)^{\frac{1}{2}}}\right)e^{-(a+b)/t}I_\nu\left(\frac{2(ab)^{\frac{1}{2}}}{t}\right)dt \doteqdot \frac{1}{x}J_\nu(2a^{\frac{1}{2}}x)J_\nu(2b^{\frac{1}{2}}x)$$

(*g*) Similarly, we get, when $k = -1$

$$-p\int_0^\infty J_0(2(pt)^{\frac{1}{2}})I_\nu\left(\frac{2(ab)^{\frac{1}{2}}}{t}\right)t^{-1}e^{-(a+b)/t}dt \doteqdot \frac{1}{x}\,J_\nu\left(2\left(\frac{a}{x}\right)^{\frac{1}{2}}\right)J_\nu\left(2\left(\frac{b}{x}\right)^{\frac{1}{2}}\right).\quad \mathbf{R}(\nu) > -\tfrac{1}{2}$$

(*h*) Let $$F(p) = ap^{-\mu+\frac{1}{2}}K_{2\nu}(ap^{\frac{1}{2}}) \doteqdot \frac{1}{x^\mu}e^{-a^2/(8x)}W_{\mu,\nu}\left(\frac{a^2}{4x}\right)U(x).$$

$$\mathbf{R}(p) > 0,\ a > 0,\ \mu \pm \nu > -\tfrac{1}{2}.$$

Then when $k = 2$ $$f(x^2) = \frac{1}{x^{2\mu+1}}e^{-a^2/(8x^2)}W_{\mu,\nu}\left(\frac{a^2}{4x^2}\right)$$

Hence by the theorem, we get,

$$\frac{p}{(2\pi)^{\frac{1}{2}}}\int_0^\infty e^{-p^2/(8t)}D_{-1}\left(\frac{p}{(2t)^{\frac{1}{2}}}\right)t^{-\mu-\frac{1}{2}}K_{2\nu}(at^{\frac{1}{2}})dt \doteqdot a^{-1}x^{-2\mu-1}e^{-a^2/8x^2}W_{\mu,\nu}\left(\frac{a^2}{4x^2}\right)U(x).$$

(i) Similarly by the theorem, when $k=-1$

$$-p\int_0^\infty J_0(2(pt)^{\frac{1}{2}})t^{-\mu-\frac{1}{2}}K_{2\nu}(a(t)^{\frac{1}{2}}dt) \doteqdot \frac{1}{a}x^{\mu-1}e^{-a^2x/8}W_{\mu,\nu}\left(\frac{a^2x}{4}\right) \quad (\mu\mp\nu)<\tfrac{1}{2}.$$

5. **Theorem. 2.** The definition Integral in this case we take to be

$$L_{\text{II}}'\{fU\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-px-qy}f(x,y)U(x,y)dxdy$$

(i) *Let* $F(p_1, p_2) \doteqdot f(x_1, x_2)U(x_1, x_2)$,

where $L_{\text{II}}^2 f\{U\}$ *is absolutely convergent in a pair of associated half planes* H_{p_1}, H_{p_2} *for* $i=1,2$.

(ii) $e^{-\lambda_i\psi_i(p_i)} \doteqdot h_i(\lambda_i, x_i)U(x_i), \quad i=1,2 \quad \lambda_i>0$

and $L_{\text{II}}\{h_i, u_i\}$ *is absolutely convergent in the half planes* D_{p_i} (say), *defined by* $\mathbf{R}(p_i)\geq 0$ *and* $\psi_i(p_i)$ *is bounded and integrable in* $(0,\infty)$. *Then*

$$K(p_1,p_2) = \frac{p_1p_2}{(\log a)^2}\int_0^\infty\int_0^\infty h_1(p_1,x_1)h_2(p_2,x_2)\frac{F(x_1,x_2)}{x_1x_2}dx_1dx_2$$

$$\doteqdot f\{a^{\varphi_1(x_1)}, a^{\varphi_2(x_2)}\}\varphi_1'(x_1)\varphi_2'(x_2), \quad a>0$$

It is sufficient if $h_1(p_1,x_1)h_2(p_2,x_2)\dfrac{F(x_1,x_2)}{x_1x_2}$ is absolutely convergent in $(0,\infty)$ in each of the variables and $f\{a^{\varphi_1(x_1)}, a^{\varphi_2(x_2)}\}\varphi_1'(x_1)\varphi_2'(x_2)$ is bounded and integrable in $(0,\infty)$ in each of the variable x_1 and x_2.

The proof is on the same lines as before. The condition (v) of Theorem 1 becomes $e^{\varphi_i(x_i)}$ tends to zero as x_i tends to $-\infty$ and to ∞ as x_i tends to ∞.

The theorem is established under conditions similar to those of theorem 1.

(Chatterjee's theorem 1 on p.24 (Thesis) is a particular case of this theorem when we put $a=e$)

6. **Theorem 3.** (i) *Let* $e^{-p_1\log a-p_2\log b}F(p_1,p_2) \doteqdot f\{x_1-\log a, x_2-\log b\}U(x_1,x_2)$
$(a, b>1)$

where $L^2_{\text{II}}\{f, U\}$ *is absolutely convergent in a pair of associated half planes* H_{p_1} *and* H_{p_2} *for* $i=1,2$.

(ii) $e^{-\lambda_i\psi_i(p_i)} = h_i(\lambda_i, x_i)$

where $\lambda_i>0$, $L_{\text{II}}^r\{f, U\}$ *is absolutely convergent in half planes* D_{p_i} (say) *defined by* $\mathbf{R}(p_i)>0$ *and further let*

$$\int_0^\infty\int_0^\infty e^{-p_1x_1-p_2x_2}f\{x_1-\log a, x_2-\log b\}dx_1dx_2$$

be convergent uniformly and a continuous function of p_1, p_2 *for values of* p_1, p_2 *in* H_{p_1}, H_{p_2}.

Then,
$$K(p_1, p_2) = p_1p_2\int_0^\infty\int_0^\infty h_1(p_1, x_1)h_2(p_2, x_2)dx_1dx_2\frac{d^2}{dx_1dx_2}\left\{e^{-x_1\log a-x_2\log b}\frac{F(x_1,x_2)}{x_1x_2}\right\}$$
$$\doteqdot f\{\varphi_1(x_1)-\log a,\ \varphi_2(x_2)-\log b\}\varphi_1'(x_1)\varphi_2'(x_2).$$

It is sufficient if $h_1(p_1, x_1)h_2(p_2, x_2)\dfrac{d^2}{dx_1dx_2}\left\{e^{-x_1\log a-x_2\log b}\dfrac{F(x_1, x_2)}{x_1x_2}\right\}$ is absolutely convergent in $(0, \infty)$ in each of the variables and
$$f\{\varphi_1(x_1)-\log a,\ \varphi_2(x_2)-\log b\}\varphi_1'(x_1)\varphi_2'(x_2)$$
is bounded and integrable in $(0, \infty)$ in each of the variable x_1 and x_2

Proof. Let us consider the integral
$$I = p_1p_2\int_0^\infty\int_0^\infty e^{-p_1x_1-p_2x_2}f\{\varphi_1(x_1)-\log a,\ \varphi_2(x_2)-\log b\}\varphi_1'(x_1)\varphi_2'(x_2)dx_1dx_2$$
$$\doteqdot f\{\varphi_1(x_1)-\log a,\ \varphi_2(x_2)-\log b\}\varphi_1'(x_1)\varphi_2'(x_2)$$

If we put $s_i = \varphi_i(x_i),\quad x_i = \varphi^{-1}(s_i) = \psi_i(s_i)$,

then
$$I = p_1p_2\int_0^\infty\int_0^\infty e^{-p_1\psi_1(s_1)-p_2\psi_2(s_2)}f\{s_1-\log a,\ s_2-\log b\}ds_1ds_2$$
$$= p_1p_2\int_0^\infty\int_0^\infty f(s_1-\log a,\ s_2-\log b)ds_1ds_2s_1s_2\int_0^\infty\int_0^\infty e^{-s_1x_1-s_2x_2}h_1(p_1, x_1)h_2(p_2, x_2)dx_1dx_2$$
$$= p_1p_2\int_0^\infty\int_0^\infty h_1(p_1, x_1)h_2(p_2, x_2)\frac{d^2}{dx_1dx_2}\left\{\int_0^\infty\int_0^\infty f(s_1-\log a,\ s_2-\log b)\right.$$
$$\left.e^{-s_1x_1-s_2x_2}ds_1ds_2dx_1dx_2\right\}$$
$$= p_1p_2\int_0^\infty\int_0^\infty h_1(p_1, x_1)h_2(p_2, x_2)\frac{d^2}{dx_1dx_2}\left\{e^{-x_1\log a-x_2\log b}\frac{F(x_1, x_2)}{x_1x_2}\right\}dx_1dx_2.$$

The change in the orders of integrations being justifiable by the conditions of the theorem.

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DISTRIBUTION-THEORETIC OBSERVATION ON THE BORN SCATTERING FORMULA (FIRST APPROXIMATION)

BY

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It is of interest to note what type of scattering potentials $V(r)$ would give *mathematically meaningful results* for the Born formula for scattering (in the first approximation). Schwartz' theory of distributions a very suitable means to this end.

We consider the scattering of a beam of particles by a field $V(r)$. We have to solve the wave equation

$$\nabla^2\psi+[k^2-U(r)]\psi=0$$

where $k^2=8\pi^2mE/h^2$, $U(r)=8\pi^2mV(r)/h^2$ and where ψ must have the asymptotic form

$$\psi\sim e^{ikz}+r^{-1}e^{ikr}f(\theta) \tag{2}$$

i.e., the solution ψ of the wave equation must represent at a large distance from the scatterer an incident wave, e^{ikz} and a scattered wave, $r^{-1}e^{ikr}f(\theta)$.

ψ satisfies the integral relation (Mott and Massey, 1949, p. 117)

$$\psi\sim e^{ikz}-r^{-1}e^{ikr}\cdot\frac{1}{4\pi}\int\exp\,[-ik(\mathbf{n}\cdot\mathbf{r}')]U(r')\psi(\mathbf{r}')d\tau' \tag{3}$$

where $\mathbf{n}$ is the unit normal in the direction of $\mathbf{r}$, so that

$$\mathbf{n}=(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$$

To obtain the first approximation for $f(\theta)$ we assume that the wave is not much diffracted by the scattering centre. We may then replace $\psi(\mathbf{r}')$ in the integral (3) by the unperturbed wave function $e^{ikz'}$. The approximation is only valid for fast particles. Thus we have from (2) and (3), by dropping the dashes

$$f(\theta)=-\frac{1}{4\pi}\int\exp\,[ik(\mathbf{n}_0-\mathbf{n})\cdot\mathbf{r}]U(r)d\tau \tag{5}$$

where $\mathbf{n}_0$ is the unit vector along the z-axis (Mott and Massey, 1949, pp. 116-7).

It is at this point that we bring in the theory of distributions to interpret the integral on the right hand side of (5). In the important case of scattering by the Coulomb field this integral does not exist in the ordinary sense. Wentzel (1927) and

Bethe (1950) brought in additional physical assumptions to give a meaningful result. The result was given mathematical justifications without bringing in any physical assumption by the methods of summability of divergent integrals by the author (Ghosh, 1947, 1948, 1949, 1956). This method also applies to a range of power laws other than the Coulombian one. We shall now interpret the right hand side of (5) in a *distributional* sense. If $f(\theta)$ be a distribution, then in general $|f(\theta)|^2$ is not defined (Schwartz, 1954). So only those $U(r)$ are permissible which give a meaningful $|f(\theta)|^2$.

The right hand side of (5) may be written as

$$-\frac{1}{4\pi}\int \exp\left[-2\pi i\mathbf{q}\cdot\mathbf{r}\right]U(r)d\tau \tag{6}$$

where $2\pi\mathbf{q} = k(\mathbf{n}-\mathbf{n}_0)$. so that $2\pi q = k\,|\,\mathbf{n}-\mathbf{n}_0\,| = 4\pi\sin\frac{1}{2}\theta/\lambda$,

$\lambda = 2\pi/k = h/mv$. The integral in (6) is the Fourier transform $\mathbf{F}U(r)$ of $U(r)$.

Let us now examine $\mathbf{F}r^m$ various values of m (real or complex) (Schwartz, 1951, p. 113-4)

$$\mathbf{F}r^m = \frac{1}{\pi^{3/2+m}}\frac{\Gamma(\frac{1}{2}m+3)}{\Gamma(-\frac{1}{2}m)}q^{-(m+3)} \tag{7}$$

for all m other than $2h$, $-3-2h$, h, integer ≥ 0 When $\mathbf{Re}(m) > 0$ or < -3, r^m is to be understood in the sense of a *pseudofunction* (or *partie finie*) [Schwartz, 1950, p. 38-40]. For the exceptional values of m

$$\mathbf{F}r^{2h} = \left(-\frac{\nabla^2}{4\pi^2}\right)^h\delta \tag{8}$$

$$\mathbf{F}\left(\mathrm{Pf.}\frac{1}{r^{3+2h}}\right) = \frac{\pi^{3/2+2h}}{\Gamma(\frac{3}{2}+h)}\,2\,\frac{(-1)^h}{h\,!}\,q^{2h}\left[\ln\frac{1}{\pi q}+\tfrac{1}{2}\left(1+\tfrac{1}{2}\ldots+\frac{1}{h}-\gamma\right)+\tfrac{1}{2}\frac{\Gamma'(\frac{3}{2}+h)}{\Gamma(\frac{3}{2}+h)}\right] \tag{9}$$

where γ is Euler's constant; the sum $1+\frac{1}{2}+\ldots+1/h$ has to be replaced by 0 for $h = 0$.

In particular for $m = 3$,

$$\mathbf{F}\left(\mathrm{Pf.}\frac{1}{r^3}\right) = \frac{2\pi^{3/2}}{\Gamma(\frac{3}{2})}\left[\ln\frac{1}{\pi q}-\tfrac{1}{2}\gamma+\tfrac{1}{2}\,\frac{\Gamma'(\frac{3}{2})}{\Gamma(\frac{3}{2})}\right] \tag{10}$$

$$\mathbf{F}\left(\ln\frac{1}{r}\right) = \frac{\Gamma(\frac{3}{2})}{2\pi^{3/2}}\left(\mathrm{Pf.}\frac{1}{q^3}\right)+\left(\frac{\gamma}{2}-\tfrac{1}{2}\cdot\frac{\Gamma'(\frac{3}{3})}{\Gamma(\frac{3}{2})}+\ln\pi\right)\delta \tag{11}$$

In the case of Coulomb potential $m = 1$ and we get at once the usual result from (7).

Now we see at once from (7)-(11) that $|f(\theta)|^2$ would be meaningful in all cases except when

$$U(r) = Cr^{2h}$$

and

$$U(r) = C \ln r, \quad (C = \text{const.}).$$

Thus *if the scattering force varies directly as any odd power of the distance or varies as inverse distance the Born scattering formula would lose all meaning.* These two exceptions arise because in these two cases $|f(\theta)|^2$ would involve products of two singular functions (distributions) which do not make any definite sense.

Another interesting case to be noted is the case of

$$U(r) = Cr^{-2}$$

In this case, the exact series solution cannot be determined because of the nature of the singularity at the origin and the indeterminacy of phase (*cf.*, Mott and Massey, 1949, p. 40-1); but on interpretation of $\mathbf{F}U(r)$ in the *distributional* sense, the Born scattering formula gives a meaningful result.

We may also note here that $\delta^{(2k)}(r)$, Pf. r^{-6-k}, Pf $r^{-6-k} \ln r$, Pf. $r^{-6-k} \ln^2 r$, etc. have Fourier transforms whose squares are defined in the ordinary sense [Gelfand i Shilov, 1958, p. 225]. Thus these potentials would also give a meaningful expression for the Born scattering cross-section.

When in an atomic field the charge density is brought in to calculate the field and thence the atomic scattering factor similar considerations apply.

In the relativistic case also the above considerations apply since in this case also the square of the absolute value of the Fourier transform of the potential is involved in the expression for the scattering cross-section (*cf.* Mott and Massey, 1949, p. 120).

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BOOK REVIEW

Pitt, H. R.,—Tauberian Theorems: Tata Institute of Fundamental Research Monographs on Mathematics and Physics, No. 2, Oxford University Press, Calcutta (India), 1958—174 pp. Rs. 22·50.

This is a treatise written by a specialist for those who are devoted to the study of Tauberian theorems and their applications in the theory of primes. There are six chapters on the following topics.

Chapter I.—Introduction, Abelian transformations, Tauberian theorems, Mercerian theorems, Miscellaneous Tauberian theorems and the prime number theorem.

Chapter II.—Elementary Tauberian theorems, Tauberian conditions, Elementary general Tauberian theorems.

Chapter III.—Classical Tauberian Theorems, Transformation of the kernel, Cesàro and Riesz summation, Abel summation, Borel summation.

Chapter IV.—Wiener's Theory, Fourier integrals, Wiener's Tauberian theorems Some extensions of Wiener's theorems, Applications to special Tauberian theorems.

Chapter V.—Mercerian theorems, Fourier-Stieltjes transforms, Geneaal Mercerian theorems, Some special Mercerian theorems, Further extensions and applications to integro-differential equations.

Chapter VI.—Tauberian Theorems and the Prime Number Theorems. The Landau-Ikehara theorem, Classical proofs of the prime number theorem, The elementary proof of the prime number theorems and related Tauberian theorems.

The manner of exposition is very stimulating. The bibliography at the end of the book will be very useful to the active workers in the field.

The production of the book is excellent and the editors of the series are to be congratulated for their success.

M. DUTTA.

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